Lecture 3

Structural Induction: A Rough Guide

Identifying structural induction problems:

1. Identify the recursive structure of the problem. Are you looking at WFFs? Parse Trees? etc...

2. Identify each recursive appearance of the structure inside its definition. Where in the structure’s definition is the recursion?

3. Divide the cases into: those without recursive appearances (base cases) and those with (inductive cases).

Proving structural induction problems:

1. Give a name for the property, i.e. $P(\varphi)$ is the property <insert property you’re proving here>

2. Base Case: Prove the base case(s), as identified in the part above.

3. Induction Step: Inductive Hypothesis and proofs for all the inductive cases.

4. Concluding statement

Semantics

The semantics of a language describes how one should interpret the language. In propositional logic, the semantics is compositional; the meaning of the formula is derived from the meanings of its parts (propositional variables and connectives).

Truth Valuation:

A truth valuation is a function with the set of all proposition symbols as domain and \{F, T\} as range.

(Symbolically, a function $t : P \mapsto \{F, T\}$.)

In other words, a truth valuation assigns a value to every propositional variable. This is where propositional variables get their meaning from.

Compound Formulas:

Let $\alpha$ and $\beta$ be two formulas that express propositions $a$ and $b$. Intuitively, we give the following meanings to combinations:

- $\neg \alpha$ Not $a$
- $\alpha \land \beta$ $a$ and $b$
- $\alpha \lor \beta$ $a$ or $b$
- $\alpha \rightarrow \beta$ If $a$, then $b$
- $\alpha \leftrightarrow \beta$ $a$ if and only if $b$

Formally, a connective represents a function from truth values to truth values.
Truth Tables

The connective ¬ is unary; it maps one value to one value.

We can show its function in a picture, known as a truth table:

<table>
<thead>
<tr>
<th>α</th>
<th>(¬α)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

The binary connectives:

<table>
<thead>
<tr>
<th>α</th>
<th>β</th>
<th>(α ∧ β)</th>
<th>(α ∨ β)</th>
<th>(α → β)</th>
<th>(α ↔ β)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

'Or' in this case is inclusive.

Alternatively, we can use valuation trees.

Truth Valuation of a Formula

Fix a truth valuation t.

Every formula α has a value under t, denoted α^t, determined as follows.

1. p^t = t(p).
2. (¬α)^t = { T if α^t = F
               F if α^t = T

3. (α ∧ β)^t = { T if α^t = β^t = T
                F otherwise

4. (α ∨ β)^t = { T if α^t = T or β^t = T
                F otherwise

5. (α → β)^t = { T if α^t = F or β^t = T
                F otherwise

6. (α ↔ β)^t = { T if α^t = β^t
                F otherwise

The value of a formula comes from the values of its variables, combined as given by its connectives.

The valuation t is necessary.

Without a valuation, a formula has no value.

Tautology, Satisfaction, Contradiction

A formula α is a tautology if and only if for every truth valuation t, α^t = T.

A formula α is a contradiction if and only if for every truth valuation t, α^t = F.

A formula α is satisfiable if and only if there is some truth valuation t such that α^t = T.
Equivalence

Formulas $\alpha$ and $\beta$ are called *equivalent* formulas,

$$\alpha \equiv \beta$$

If and only if $\alpha$ and $\beta$ have the same truth value under any valuation (meaning they have the same final column in their truth tables).

In symbols: $\alpha^t = \beta^t$, for every valuation $t$.

*Lemma*: Suppose that $\alpha \equiv \beta$. Then for any formula $\gamma$, and any connective $\star$, the formulas $(\alpha \star \gamma)$ and $(\beta \star \gamma)$ are equivalent:

$$(\alpha \star \gamma) \equiv (\beta \star \gamma)$$