Lecture 9

Soundness and Completeness

Let $S$ be a proof system.

**Soundness**: $S$ is sound if every proof in $S$ is also entailed.

\[ I f \Sigma \vdash_S \phi then \Sigma \models \phi. \]

Every proof in $S$ is a logical consequence (is "true").

This gives us a method for telling that an argument does not have a proof in the system: show that the conclusion is not entailed by the premises.

**Completeness**: $S$ is complete if every entailment has a proof in $S$.

\[ I f \Sigma \models \phi then \Sigma \vdash_S \phi. \]

Every logical consequence has a proof in $S$.

This gives us a method for telling that an argument has a proof in the system: show that the conclusion is entailed by the premises.

*An Example of Using Soundness*

Show that $\{(p \lor q)\} \not\vdash_{ND} p$.

**Proof**: By the contrapositive of soundness.

The contrapositive of soundness in this instance is the statement:

\[ if \{(p \lor q)\} \not\models p then \{(p \lor q)\} \not\vdash_{ND} p \]

It suffices then to show that the entailment does not hold.

Let $t$ be the valuation such that $p^t = F$ and $q^t = T$. Then $(p \lor q)^t = T$.

By the definition of semantic entailment, since there is a truth valuation such that $(p \lor q)^t = T$ and $p^t = F$ then $\{(p \lor q)\} \not\models p$.

By soundness, it follows that $\{(p \lor q)\} \not\vdash_{ND} p$. 

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**Theorem**: Natural Deduction is sound for propositional logic.

*Soundness* of Natural Deduction means that the conclusion of a Natural Deduction proof is always a logical consequence of the premises:

\[ \text{If } \Sigma \vdash_{ND} \alpha, \text{ then } \Sigma \models \alpha. \]

**Proof**: By induction of on the length of the proof (including partial proofs).

Suppose that a formula \( \alpha \) appears at line \( n \) of a partial deduction, which may have one or more open sub-proofs. Let \( \Sigma \) be the set of premises used and \( \Gamma \) be the set of assumptions of open sub-proofs. Then \( \Sigma \cup \Gamma \models \alpha \).

**Base case**: The shortest deductions have length 1: it is a premise or an assumption. (Exercise)

**Inductive Hypothesis**: the property holds for each \( n \leq k \);

**Inductive Step**: prove the property holds for \( k + 1 \). The case that \( \alpha' \) is an assumption or premise is trivial. If it is not one of these, then formula \( \alpha' \) must have a justification by some inference rule. One must consider each possible rule.

**Case I**: \( \alpha' \) was justified by *Reflexivity*. Exercise.

**Case II**: \( \alpha' \) was justified by \( \land i \).

Rule \( \land i \) requires that \( \alpha' = (\alpha_1 \land \alpha_2) \), where each of \( \alpha_1 \) and \( \alpha_2 \) appear at steps \( m_1 \) and \( m_2 \), respectively, where \( m_1, m_2 \leq k \). Also, any sub-proof open at steps \( m_1, m_2 \) is still open at step \( k + 1 \).

The induction hypothesis applies to both \( \alpha_1, \alpha_2 \): \( \Sigma \cup \Gamma \models \alpha_1 \) and \( \Sigma \cup \Gamma \models \alpha_2 \).

By the definition of semantic entailment, \( \Sigma \cup \Gamma \models \alpha' \) as required.

*Exercise*. Let \( \Sigma \models \alpha_1 \) and \( \Sigma \models \alpha_2 \). Prove that \( \Sigma \models (\alpha_1 \land \alpha_2) \)

**Case III**: \( \alpha' \) was justified by \( \land e \). Exercise.

**Case IV**: \( \alpha' \) was justified by \( \rightarrow i \).

Rule \( \rightarrow i \) requires that \( \alpha' = (\alpha_1 \rightarrow \alpha_2) \) and a closed sub-proof starting with assumption \( \alpha_1 \) at line \( j \) and ending with conclusion \( \alpha_2 \) by step \( k \) (Note: \( j \leq k \)). Also, any additional sub-proofs open (excluding the current sub-proof) when the assumption of \( \alpha_1 \) is made, is still open at step \( k + 1 \).

The induction hypothesis thus implies \( \Sigma \cup (\Gamma \cup \{\alpha_1\}) \models \alpha_2 \).

Hence \( \Sigma \cup \Gamma \models (\alpha_1 \rightarrow \alpha_2) \), as required.

*Exercise*. Let \( \Sigma \cup \{\alpha_1\} \models \alpha_2 \). Prove that \( \Sigma \models (\alpha_1 \rightarrow \alpha_2) \).

**Case V**: \( \alpha' \) was justified by \( \rightarrow e \). Exercise.
Case VI: $\alpha'$ was justified by $\bot i$.

Rule $\bot i$ requires that $\alpha'$ be the pseudo-formula $\bot$, and that the proof contains formulas $\alpha_1$ and $\neg\alpha_1$, at steps $m_1$ and $m_2$, respectively, where $m_1, m_2 \leq k$.

Also, any sub-proof open at steps $m_1, m_2$ is still open at $k + 1$.

By the induction hypothesis, both $\Sigma \cup \Gamma \vDash \alpha$ and $\Sigma \cup \Gamma \vDash (\neg \alpha)$.

Thus $\Sigma \cup \Gamma$ is unsatisfiable, and $\Sigma \cup \Gamma \vDash \alpha'$ for any $\alpha'$.

Exercise. Let $\Sigma$ be unsatisfiable. Prove that $\Sigma \vDash \alpha$ for any $\alpha$.

Case VII: $\alpha'$ was justified by $\bot e$. Exercise.

Case VIII: $\alpha'$ was justified by $\forall i$. Exercise.

Case IX: $\alpha'$ was justified by $\forall e$. Exercise.

Case X: $\alpha'$ was justified by $\neg i$. Exercise.

Case XI: $\alpha'$ was justified by $\neg \neg e$. Exercise.

By induction, the property holds for any $k$. This completes the proof of soundness.
**Theorem:** Natural Deduction is complete for propositional logic.

Completeness of Natural Deduction means that all logical consequences are provable in Natural Deduction.

If \( \Sigma \vdash_{ND} \alpha \), then \( \Sigma \vDash \alpha \).

*For the purposes of this proof, assume \( \Sigma \) to be a finite set of premises.*

**Proof idea:**

1. Logical consequences can be expressed as tautologies.
2. All tautologies are provable in Natural Deduction.
3. A ND proof of a tautology can be transformed into a proof from the original set of premises in the logical consequence to its conclusion.

**Proof:** Let \( \Sigma = \{ \alpha_1, \alpha_2, ..., \alpha_n \} \) for well-formed formulas \( \alpha_i \) (for all \( 1 \leq i \leq n \)).

**Lemma 1:** If \( \Sigma \vDash \beta \), then \( \emptyset \vDash (\alpha_1 \rightarrow (\alpha_2 \rightarrow (... \rightarrow (\alpha_n \rightarrow \beta)...)) \).  

**Proof:** Exercise.

**Lemma 2:** For any well-formed formula \( \gamma \), if \( \emptyset \vDash \gamma \), then \( \emptyset \vdash_{ND} \gamma \).

For a tautology, every line of its truth table ends with \( T \).
We can mimic the construction of a truth table using inferences in ND.

**Proof of Lemma 2:** Let \( \gamma \) be any formula containing atoms \( p_1, ..., p_n \). Let \( t \) be a valuation.

To prove this, we first need the following sublemma:

**Sublemma:** For any formula \( \gamma \) containing atoms \( p_1, ..., p_n \) and any valuation \( t \), and define \( \hat{p}_1, ..., \hat{p}_n \) as

\[
\hat{p}_i = \begin{cases} 
  p_i & \text{if } p_i^t = T \\
  \neg p_i & \text{if } p_i^t = F 
\end{cases}
\]

then,

- If \( \gamma^t = T \) then \( \{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash_{ND} \gamma \)
- If \( \gamma^t = F \) then \( \{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\} \vdash_{ND} (\neg \gamma) \)

This sublemma is proved by structural induction on the formulas.  
**Proof:** Exercise.  
**Hint:** You will need to consider all forms of \( \gamma \) and whether it is \( T \) or \( F \) under \( t \).

Now we can construct a proof of a tautology via \( 2^n \) subproofs of the possible combinations of these atoms with their negations:

1. Start with \( n \) lines of Law of Excluded Middle, one for each atom.

2. Then construct a nested subproof: first with the assumption of \( p_1 \), then another subproof with the assumption \( p_2, ..., \) then lastly a subproof with the assumption \( p_n \). Once the subproof starting with assumption \( p_n \) yields \( \gamma \), close it and open another subproof with the assumption \( (\neg p_n) \). This will also yield \( \gamma \). After closing this subproof, use \( \lor e \) on the corresponding line of LEM and the two subproofs \( \gamma \).
3. Repeat to enumerate through all $2^n$ subproofs. Each fully nested subproof (a subproof with the open assumption of either $p_i$ or $\neg p_i$ for all $i \leq n$) corresponds to a line of a truth table and will prove $\gamma$ (this follows from the sublemma).

The proof will look like:

\begin{align*}
1. & (p_1 \lor \neg p_1) \quad \text{L.E.M.} \\
2. & (p_2 \lor \neg p_2) \quad \text{L.E.M.} \\
: & : \\
n. & (p_n \lor \neg p_n) \quad \text{L.E.M.} \\
n + 1. & p_1 \quad \text{assumption} \\
& \quad \vdots \\
& \gamma \\
& \quad \vdots \\
& \neg p_2 \quad \text{assumption} \\
& \quad \vdots \\
& \gamma \\
m. & \gamma \quad \text{\textit{\&}\textit{e}: 2, \ldots} \\
m + 1. & \neg p_1 \quad \text{assumption} \\
& \quad \vdots \\
& \vdots \\
& \gamma \quad \text{\&}\textit{e}: m + 1, \ldots \\
\ell. & \gamma \quad \text{\&}\textit{e}: 1, m - (n + 1), \ell - (m + 1) \\
\end{align*}

Where $p_i$ is a variable in $\gamma$ and each is assumed true or false, the previous sublemma provides a proof.

**Lemma 3:** If $\emptyset \vdash_{ND} (\alpha_1 \rightarrow (\alpha_2 \rightarrow (\ldots \rightarrow (\alpha_n \rightarrow \beta)\ldots)))$, then $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \vdash_{ND} \beta$.

**Proof of Lemma 3:** Transform the proof of $\emptyset \vdash_{ND} (\alpha_1 \rightarrow (\alpha_2 \rightarrow (\ldots \rightarrow (\alpha_n \rightarrow \beta)\ldots)))$ into a proof of $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \vdash_{ND} \beta$

\begin{align*}
1. & \alpha_1 \quad \text{Premise} \\
2. & \alpha_2 \quad \text{Premise} \\
: & : \\
n. & \alpha_n \quad \text{Premise} \\
: & : \\
n + m. & (\alpha_1 \rightarrow (\alpha_2 \rightarrow (\ldots \rightarrow (\alpha_n \rightarrow \beta)\ldots))) \quad \text{The proof of tautology} \\
n + m + 1. & (\alpha_2 \rightarrow (\ldots \rightarrow (\alpha_n \rightarrow \beta)\ldots))) \quad \text{Last line of the proof} \rightarrow \text{e: } 1, n + m \\
n + m + 2. & (\alpha_3 \rightarrow (\ldots \rightarrow (\alpha_n \rightarrow \beta)\ldots))) \quad \rightarrow \text{e: } 2, n + m + 1 \\
: & : \\
2n + m. & \beta \rightarrow \text{e: } n, 2n + m - 1 \\
\end{align*}

This completes the proof of completeness.