Logic and Computation - Introduction

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with thanks to Anna Lubiw
Logic - Introduction

- What is logic?
- History of logic - beginnings
- Why study logic?
- Applications of logic to computer science
What is logic?

Logic is the **Science of Reasoning**

- **Etymology:** *Logykos* (Greek) - pertaining to reasoning
- **Logic** = The science of reasoning, proof, thinking, or inference
- **Logic** = The fundamental science of thoughts and its categories
- **Logic** = The science or art of reasoning as applied to a department of knowledge
- **Logic** = The analysis and appraisal of arguments
- We all do logic when we try to clarify reasoning and separate good from bad reasoning
Logic - Introduction

- What is logic?
- History of logic - the beginnings
- Why study logic?
- Applications of logic to computer science
History of logic - the beginnings

- Aristotle (Greek philosopher, 384-322 B.C.) is the first person to offer the outlines of a comprehensive systems for codifying and evaluating a very wide range of arguments and reasoning.
Aristotle - “Father of Logic”

- **Aristotle**: Student of Plato ("Father of Western Philosophy")
- **Aristotle**: Tutor of **Alexander the Great** (356 - 323 B.C.)

For Aristotle, logic is the instrument by means of which we come to know anything

Aristotle wrote the earliest known formal study of logic
Aristotelian logic

- Formalizes the basic principles of good reasoning, and provides a way to evaluate specific cases of reasoning
- A syllogism is a kind of logical argument in which one proposition (the conclusion) is inferred from two or more others (the premises) of a specific form.
- The following is an example of an Aristotelian syllogism:
  - All humans are mortal.
  - Socrates is human.
  - ___________________
  - Socrates is mortal.
- The horizontal line separates the premises from the conclusion.
- This syllogism is an example of good reasoning - constitutes a good argument - because it is truth-preserving.
- That is, if the first two sentences (premises) are true, then the third sentence (conclusion) must also be true.
Aristotelian logic

- Correctness depends on form (structure) not content
- All Accords are Hondas.
  All Hondas are Japanese.
  All Accords are Japanese.
  has the form of hypothetical syllogism
- All $x$ are $y$.
  All $y$ are $z$.
  All $x$ are $z$.
- Is this a correct argument?
  All $x$ are $y$.
  Some $y$ are $z$.
  Some $x$ are $z$. 
Logic - Introduction

- What is logic?
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Why study logic?

- Most people find logic enjoyable
- Logic improves one’s general powers of analytical thinking
- Logic is fundamental to Computer Science

“I expect that digital computing machines will eventually stimulate a considerable interest in symbolic logic . . . . The language in which one communicates with these machines . . . forms a sort of symbolic logic.” (Alan Turing, 1947)
What is logic?
History of logic - beginnings
Why study logic?
Applications of logic to computer science
Applications of logic to computer science

- The underpinnings of all electronic computers are logic gates

  ![Logic Gates Diagram]

- Electronic digital circuits are formed out of logic gates

- Logic can be used for the minimization of the number of components of electronic circuits

![Electronic Circuit Image]
Applications of logic to computer science

- **Artificial Intelligence** - Expert systems (knowledge base + inference engine)
  - **DENDRAL** (Stanford University, 1960s) - an expert system to aid the identification of unknown organic molecules.
  - **MYCIN** (Stanford University, 1972), an expert system for treating blood infections *(medical diagnosis systems: The user describes their symptoms to the computer, as they would to a human doctor, and the computer returns a medical diagnosis.)* It gave acceptable therapy in about 69% of cases, which was roughly the same level of competence as human specialists in blood infections and rather better than general practitioners.
  - **MISTRAL** (Italy, 1990s) - an expert system for monitoring dam safety (still operational).
Applications of logic to computer science

- Automated theorem proving; automated proof verifiers
- **Databases**: core of modern database systems (e.g. SQL) uses first-order logic
- **Programming**:
  - Program specification
  - **Formal verification** (how do we know that a program does what it is supposed to do)
Why do we need formal verification?

- Errors occur at the specification, design, and implementation levels of computer systems
- Errors are often costly and dangerous
- Hardware example:
  * Intel’s Pentium FDIV bug (1994) cost them half a billion dollars. Intel is now heavily invested in formal verification.
- Software examples:
  * security flaws in Microsoft, Linux, Adobe, . . .
  * death of cancer patients who received severe overdoses of radiation due to a software error in the Therac 25 (1985 – 1987)
Applications of logic to computer science

- The programming language **PROLOG** (PROgramming with LOGic)
- DNA Computing
- Synthetic biology
Summary

- Logic has a long history and many applications
- We all use logic in our everyday life
- We will start with propositional logic
What is a logical argument?
Some important logical arguments
Propositions
Logical connectives
Logical arguments, valid or invalid

- **Logic** is the analysis and appraisal of arguments.
- An **argument** is a set of statements, namely one or several premises and a conclusion, usually connected by “therefore”.
- An argument here isn’t a quarrel or fight. Rather it is the verbal expression of a reasoning process.
- Consider this argument about the Cuyahoga River:

  No pure water is burnable.
  Some Cuyahoga River water is burnable.

  ————————————————————

  Some Cuyahoga River water is not pure.

  (the horizontal line is short for therefore.)

  The argument is **valid**.

A valid (correct, sound) argument is one in which whenever the premises are true, the conclusion is also true.
The Cuyahoga River is a river in the United States, located in Northeast Ohio, that feeds into Lake Erie. The river is famous for having been so polluted that it "caught fire" in 1969. The event helped to spur the environmental movement in the US.
Logical arguments, valid or invalid

- Logic studies **forms** of reasoning.
- The **content** might deal with anything - water purity, mathematics, cooking, nuclear physics, ethics, or whatever.
- When we learn logic, we are learning tools of reasoning that can be applied to **any** subject.
- Let us take another argument:

  No pure water is burnable.
  Some Cuyahoga River water is not burnable.

  ________________

  Some Cuyahoga River water is pure water.

  **This argument is invalid (incorrect, unsound).**
  (The whole Cuyahoga river could be polluted by non-burnables.)
Logical arguments

Example of a logical argument (A)

1. If the demand rises, then companies expand.
2. If companies expand, then companies hire workers.
   3. If the demand rises, then companies hire workers.

- This argument consists of two premises (1 and 2), and the conclusion (3).

**IMPORTANT**: One can argue against the conclusion and claim that it is wrong. However, as soon as the premises are accepted as true, the conclusion must also be true.

In such a case we say that the conclusion logically follows from the premises, or that the argument is valid (correct, sound).
Logical arguments

Example of logical argument (B)

1. This computer program has a bug, or the input is erroneous.
2. The input is not erroneous.

___________________________________________________________________________

3. This computer program has a bug.

Compound statements consist of several parts, each of which is a statement in its own right.

- In Example (A): “demand rises”, “companies expand” are connected by if, then
- In Example (B): “this computer program has a bug”, “the input is erroneous” are connected by or.
Some important logical arguments

To see which arguments are correct and which not, we abbreviate the essential statements by substituting letters $p, q, r$.

The letter $p$ may express the statement that “demand rises”,
The letter $q$ may express the statement “companies expand”,
The letter $r$ may express the statement “companies hire workers”

Then the logical argument (A) becomes:
1. If $p$ then $q$.
2. If $q$ then $r$.

3. If $p$ then $r$.

This argument is called a hypothetical syllogism.
Some important logical arguments

1. \( p \) or \( q \).
2. Not \( q \).

\[ \text{________} \]
3. \( p \).

This argument is called the **disjunctive syllogism**

1. If \( p \) then \( q \).
2. \( p \).

\[ \text{________} \]
3. \( q \).

This argument is called **modus ponens**.
Propositions

**Definition:** Any statement that is either true or false is called a proposition.

Meaningless statements, commands or questions are not propositions.

- \( p, q, r \) are called propositional variables.
- True (T) and false (F) (or 1 and 0) are propositional constants.
- Any propositional variable can be assigned the value 0 or 1.
Proposition examples

Which of the following sentences are propositions? What are the truth values of those that are propositions?

1. Waterloo is the capital of Ontario.
2. Montreal is the capital of Canada.
3. $2 + 3 = 5.$
4. $5 + 7 = 10.$
5. $x + 2 = 11.$
6. Answer this question.
7. $x + y = y + x$ for every pair of real numbers $x$ and $y$.
8. Do not pass go.
9. What time is it?
Propositions

- Propositional variables are **atomic propositions**, that is, they cannot be further subdivided.
- **Compound propositions** are obtained by combining several atomic propositions.
- The function of the words **or, and, not** is to combine propositions, and they are therefore called **logical connectives**.
Logical connectives

Statements formulated in natural languages are frequently ambiguous because the words can have more than one meaning. We want to avoid this. Therefore we introduce new mathematical symbols to take the role of connectives.

**Convention:** Stating a proposition in English implies that this proposition is true.

“it is true that cats eat fish” = “cats eat fish”.

Similarly, if \( p \) is a proposition, then “\( p \)” means “\( p \) is true” or that “\( p \) holds”.
Negation

Definition Let $p$ be a proposition. The compound proposition $\neg p$, pronounced “not $p$”, is the proposition that is true when $p$ is false, and that is false when $p$ is true.

- $\neg p$ is called the negation of $p$.
- The connective $\neg$ may be translated into English as “It is not the case that,” or simply by the word “not”.

Truth table for negation

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
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<td>0</td>
<td>1</td>
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</tbody>
</table>
Conjunction

Definition. Let $p$ and $q$ be two propositions. Then $p \land q$ is true if and only if both $p$ and $q$ are true.

- $p \land q$ is called the conjunction of $p$ and $q$.
- the connective $\land$ is pronounced “and” and may be translated by the English word “and”.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
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</table>
Observations on conjunction

- In English we often use shortcuts that are not allowed in logic statements. “He eats and drinks.” really means “He eats, and he drinks.”

- In logic, every statement must have its own subject and its own predicate! Taking now $p$: “He eats.” and $q$: “He drinks.”, our sentence becomes $p \land q$

- Sometimes we use words other than “and” to denote a conjunction such as but, in addition to, and moreover.

- Not all instances of the word “and” denote conjunctions. Example: The word “and” in “Jack and Jill are cousins” is not a conjunction at all!
**Disjunction**

**Definition.** Let $p$ and $q$ be two propositions. Then $p \lor q$ is false if and only if both $p$ and $q$ are false. If either $p$ or $q$ or both are true, then $p \lor q$ is true.

- $p \lor q$ is called the **disjunction** of $p$ and $q$
- the connective $\lor$ is pronounced “or” and can usually be translated into English by the word “or”

**Truth table for disjunction**

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
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</table>
Observations on disjunction

The English word “or” has two different meanings.

- **Exclusive or**
  “You can either have soup or salad” = soup or salad, but not both

- **Inclusive or**
  “The computer has a bug, or the input is erroneous”

- To avoid ambiguity one should translate $p \lor q$ into English as “$p$ or $q$, or both”

**Note:** When performing the disjunction of two sentences, always make sure that the sentences are complete: each sentence must have its own subject and predicate.

“There was an error on line 15 or 16” must first be expanded to “There was an error on line 15, or there was an error on line 16”
**Conditional**

**Definition.** Let $p$ and $q$ be two propositions. Then $p \rightarrow q$ is false if $p$ is true and $q$ is false, and $p \rightarrow q$ is true otherwise.

- $p \rightarrow q$ is called the **conditional** of $p$ and $q$.
- The conditional of $p$ and $q$ may be translated into English by using the “If...then” construct, as in “If $p$, then $q$”, or to “It is not the case that $p$ is true and $q$ is false”.
- $p \rightarrow q$ means that, whenever $p$ is correct, so is $q$.
- $p$ is called **antecedent**, $q$ is called **consequent**.

The truth table for conditional:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
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Observations on the conditional

Generally, if \( p \) is false, then “\( p \rightarrow q \)” is vacuously true, since in such case the verification of “if \( p \) then \( q \)” does not require doing anything to deduce \( q \) from \( p \).

Although unusual, this yields no inconsistency with everyday speech.

Example: “If the sun will rise from the West, I will eat my hat.”

My statement will never be contradicted (and in that sense it is true) because I know that “the sun will rise from the West” is false.
Equivalent ways of expressing a conditional

- The following are logically equivalent:

1. \( p \rightarrow q \).
2. If \( p \) then \( q \).
3. Whenever \( p \), then \( q \).
4. \( p \) is sufficient for \( q \).
5. \( p \) only if \( q \).
6. \( p \) implies \( q \).
7. \( q \) if \( p \).
8. \( q \) whenever \( p \).
9. \( q \) is necessary for \( p \).
10. \( q \) is implied by \( p \).
Equivalent ways of expressing a conditional

**Example:** Try understanding the equivalence of the statements from the previous slide using the example wherein

- $p$ stands for “$n$ is divisible by 6”
- $q$ stands for “$n$ is divisible by 3”.

**Notably,**

- “$p$ only if $q$” is translated as “$p \rightarrow q$”
- “$p$ if $q$” is translated as “$q \rightarrow p$”
- “$p$ is sufficient for $q$” is translated as “$p \rightarrow q$”
- “$p$ is necessary for $q$” is translated as “$q \rightarrow p$”
**Biconditional**

**Definition.** Let $p$ and $q$ be two propositions. Then $p \leftrightarrow q$ is true whenever $p$ and $q$ have the same truth values.

- The proposition $p \leftrightarrow q$ is called **biconditional** or **equivalence**, it is pronounced “$p$ if and only if $q$”,
- When writing, one frequently uses **iff** as an abbreviation for “if and only if”

**Truth table for biconditional**

<table>
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<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \leftrightarrow q$</th>
</tr>
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One should always be aware of the difference between equivalence and implication. In English, it is not always clear which connective is intended, as seen in the example below.

Eating hamburgers at a fast-food bar is equivalent to aiding the destruction of the world’s rainforest.

This sentence looks like an equivalence, but if we swap the sentence around, we can see that something is wrong.

Aiding the destruction of the rainforest is equivalent to eating hamburgers at a fast-food bar.

In fact, the intended meaning is implication:
If one eats hamburgers at a fast-food bar then one is aiding the destruction of the world’s rainforest.
Ambiguity and imprecision

Logic helps to clarify the meanings of descriptions written, for example, in English. After all, one reason for our use of logic is to state precisely the requirements of computer systems.

Descriptions in natural languages can be ambiguous or imprecise.

- An ambiguous sentence can have more than one distinct meaning.

- In contrast, an imprecise or vague sentence has only one meaning, but, as a proposition, the distinction between the circumstances under which it is true and the circumstances under which it is false is not clear-cut.
David and John from Toronto are coming for a visit. Who is from Toronto? David or John or both? It is impossible to know without further information.

I know a much funnier man than Bill. This may have two meanings: I know a much funnier man than Bill does, or I know a much funnier man than Bill is.

Don’t leave animals in cars because they rapidly turn into ovens. (From *News Quiz*, BBC Radio 4, 10 October, 1994). The immediate reading is far from the intended meaning.
Imprecise sentences: Examples

- John is tall.
  We do not know exactly what tall means. A more precise description is John is over 2 meters tall.

- This computer is fast.
  The meaning of “fast” is imprecise - fast compared to what? A more precise description would be This computer executes 2 million instructions per second.
Dealing with imprecision and ambiguity

- An **ambiguous** sentence usually has several interpretations. Ambiguity has to be eliminated by querying the author of the sentence or by examining the context.

- Imprecision or vagueness arises from the use of qualitative descriptions. Often we need to introduce some quantitative measures to remove vagueness.
Further remarks on connectives

- \( \neg \) is the only unary connective, that is, \( \neg p \) negates a single proposition.

- All other connectives are binary connectives (they require two propositions which are joined by the connective).

- The binary connectives \( \lor, \land, \leftrightarrow \) are symmetric, in the sense that the order of the two propositions joined by the connective does not affect the truth value of the resulting propositions. The truth value of \( p \land q \) is the same as the truth value of \( q \land p \).

- The connective \( \rightarrow \) is not symmetric: \( p \rightarrow q \) and \( q \rightarrow p \) have different truth values.
A riddle

Suppose the following two statements are true:

(1) I love Betty or I love Jane.

(2) If I love Betty then I love Jane.

Does it necessarily follow that I love Betty? Does it necessarily follow that I love Jane?
A puzzle: The island of knights and knaves

There is an island in which certain inhabitants, called knights always tell the truth, and others, called knaves always lie. It is assumed that every inhabitant of this island is either a knight or a knave.

Suppose A says, “Either I am a knave or B is a knight.”

What are A and B?
Translating from English into the language of propositional logic

(a) Write the truth table for “exclusive or”.
(b) Give the truth tables for $p \land p$, $p \lor p$, $p \land 1$, and $p \land 0$.
(c) Translate the following statements into logic formulas:

1. He is clever and diligent.
2. He is clever but not diligent.
3. He didn’t write the letter, or the letter was lost.
4. He must study hard, otherwise he will fail.
5. He will fail, unless he studies hard.
6. He will go home, unless it rains.
7. He will go home only if it rains.
8. If it rains, he will be at home; otherwise, he will go to the market or school.
9. The sum of two numbers is even if both numbers are even or both numbers are odd.
Write each of the following statements in the form $p \rightarrow q$.

1. It snows whenever the wind blows from the northeast.
2. The apple trees will bloom if it stays warm for a week.
3. That the Toronto Maple Leafs win the Stanley Cup implies that they beat the Montreal Canadiens.
4. It is necessary to walk 3,500 meters to get to the top of Mount Everest.
5. To get promoted to Full Professor at Waterloo, it is sufficient to win the Nobel Prize.
6. If you drive more than 800 kilometers, you will need to buy gasoline.
7. Your guarantee is good only if you bought your iPhone less than 90 days ago.