Predicate Logic - Semantics

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The language of predicate logic $\mathcal{L}^{\text{pred}}$ is a syntactic object of no semantic significance. The formulas of $\mathcal{L}^{\text{pred}}$, however, are intended to express statements. This is accomplished by interpretations.

Interpretations for the formulas in the propositional language $\mathcal{L}^p$ were simple: they consisted of value assignments of the propositional symbols.

The language $\mathcal{L}^{\text{pred}}$ includes more classes of symbols and hence the interpretations for it are more complex.

An interpretation for a formula in $\mathcal{L}^{\text{pred}}$ must contain sufficient information to determine whether the formula is true or false.
1. The logical symbols:

- The **connectives** will be interpreted as in Propositional Logic.
- The meaning of **quantifiers** has been explained intuitively.
- The equality symbol \( \approx \) denotes the predicate (relation) of equality.
- The **variable symbols** will be interpreted as variables ranging over the domain.
- **Punctuation symbols** serve just like punctuation in natural languages.
Interpretations

2. The non-logical symbols (parameters):

- In the case when $L^{pred}$ is associated with a structure, they are interpreted as the designated individuals, relations, and functions related with the structure. Accordingly, the sentences (closed formulas) are intended to express propositions about the structure.

- If $L^{pred}$ is not associated with any structure, a domain is still required for each interpretation. In such cases, however, the domain is merely an arbitrary non-empty set. Then the non-logical symbols are interpreted as arbitrary individuals, and arbitrary relations and functions on it.
• Recall that an $n$-ary relation on a domain $D$ is a subset of $D^n = D \times D \times ... D$ ($n$ times).

• Recall that the equality relation on $D$ is the subset $\{(x, y) | x, y \in D \text{ and } x = y\}$ or alternatively $\{(x, x) | x \in D\}$.

• The italic capital Latin letter $I$ (with or without subscripts and superscripts) will be used for any interpretation.
Definition (Interpretation)

An interpretation for the language of predicate logic $L^{pred}$ consists of a domain $D$ called the domain of the interpretation, and a function, denoted by $I$, with the properties:
(1) For each individual constant symbol $a$ and free variable symbol $u$, we have that $I(a), I(u) \in D$.
(2) For each relation symbol (predicate) $P$, $I(P)$ is an $n$-ary relation on $D$, that is, $I(P) \subseteq D^n$; and $I(\approx) = \{(x, x) \mid x \in D\} \subseteq D^2$.
(3) For each $m$-ary function symbol $f$, $I(f)$ is a total $m$-ary function of $D$ into $D$, that is, $I(f) : D^m \rightarrow D$.

Note: $I(a), I(u), I(P), I(\approx), I(f)$ should not be confused with $a, u, P, \approx, f$. 
Proof that Unicorns exist
(R. Smullyan, based on an argument by Descartes)
• Prove the (possibly) stronger statement there exists an existing unicorn (an existing unicorn is, of course, an unicorn that exists).
• There are exactly two possibilities:
  1. An existing unicorn exists.
  2. An existing unicorn does not exist.
• Possibility (2) clearly is contradictory.
• Thus, only possibility (1) can occur, q.e.d.
What is wrong with this proof?

- The fallacy lies in the double-meaning of the word *an*, which in some contexts means *every* (∀) and in others means *at least one* (∃):
  - “*An* owl has large eyes” means *All owls have large eyes*.
  - “*An* owl is in my house” means *There exists an owl in my house*.

- When we say “*An* existing unicorn exists” it is not clear if we mean
  (a) *All existing unicorns exist* or
  (b) *There exists at least one existing unicorn*. 
The importance of a non-empty domain

- If the domain $D$ is the set of all existing unicorns, and $E(x)$ means “$x$ exists”, then:
  - (a) All existing unicorns exist is translated as $\forall x E(x)$.
  - (b) There exists at least one existing unicorn is translated as $\exists x E(x)$.
- Note that $\exists x E(x)$ (which is what we actually wanted to prove) is false.
- However, $\forall x E(x)$ is vacuously true, because the domain $D$ is the empty set.
- **NOTE:** $\forall x P(x)$ is always vacuously true in an empty domain.
- In general, we want $\forall x P(x) \rightarrow \exists x P(x)$ to be true.
- This would also be false in an empty domain.
- This is why we defined a domain as being **non-empty**!
As an example - not correct in every detail - think of a programming language function (in the sense of, say, PASCAL).

- The specification of such a function will make a statement about the connection between the input parameters and the function value.
- There may be identifiers declared as constants and there may be global variables.
- To understand what the function does we need to know what domain we are talking about, which values the constants have and which values the global variables have.
- In our context, the global variables are modelled by free variable symbols.
Examples

Consider the sentence (closed formula, no free variables)

$$\forall x(F(x) \lor H(x) \rightarrow G(x)).$$

In one interpretation,

- The domain is $D_1 = \text{the set of all ships}$
- $I_1(F)$ is a unary predicate on $D_1$ defined by $I_1(F)(x): x$ is on fire, which takes the value 1 if $x$ is on fire and the value 0 if $x$ is not on fire.
- $I_1(H)$ is the unary predicate on $D_1$ defined by $I_1(H)(x): x$ has a hole.
- $I_1(G)$ is the unary predicate defined by $I_1(G)(x): x$ sinks.

In this interpretation the formula says:

“Every ship that is on fire or has a hole sinks.”
Another interpretation

\[ \forall x (F(x) \lor H(x) \rightarrow G(x)). \]

In the second interpretation,

- The domain is \( D_2 \) is the set of integers.
- \( I_2(F) \) is the unary predicate defined by \( I_2(F)(x) : x \) is positive.
- \( I_2(H) \) is the unary predicate defined by \( I_2(H)(x) : x \) equals zero.
- \( I_2(G) \) is the predicate defined by \( I_2(G)(x) : x \geq 0. \)

Then the formula says

“An integer that is 0 or strictly greater than 0 is greater than or equal to 0.”
The third interpretation

\[ \forall x (F(x) \lor H(x) \rightarrow G(x)). \]

In a 3rd interpretation, the domain is \( D_3 \) and the function is \( I_3 \) where:

- \( D_3 \) is the set of all animals.
- \( I_3(F) \) is the unary predicate defined by \( x \) is an unicorn.
- \( I_3(H) \) is the unary predicate defined by \( x \) is an rhinoceros.
- \( I_3(G) \) is the unary predicate defined by \( x \) is an animal with exactly one horn.

Then the formula says Every animal which is a unicorn or a rhinoceros has exactly one horn.
∀x(\(F(x) \lor H(x) \rightarrow G(x)\)).

In a 4th interpretation, the domain is \(D_4\) and the function is \(l_4\) where:

- \(D_4\) is the set of all attempted \(C^{++}\) programs.
- \(l_4(F)\) is the predicate defined by \(l_4(F)(x): x\) is a syntactically incorrect attempted \(C^{++}\) program.
- \(l_4(H)\) is the predicate defined by \(l_4(H)(x): x\) is a semantically incorrect attempted \(C^{++}\) program.
- \(l_4(G)\) is the predicate defined by \(l_4(G)(x): x\) is an incorrect attempted \(C^{++}\) program.

Then the formula says that Every attempted \(C^{++}\) program which is incorrect syntactically or semantically is incorrect.
Other examples of interpretations

- Define the predicate “x loves y” by $L(x, y)$, which is 1 if x loves y, and false otherwise.
- English sentences like “everyone loves someone” can be formalized by predicate logic formulas like $\forall x \exists y L(x, y)$.
- Using just the two quantifiers and the predicate $L$, but no logical connectives and no function or individual constant symbols, formulas with 8 different meanings can be built.
- The following diagrams show interpretations that make each of these formulas true, where the domain consists of exactly five individuals $\{a, b, c, d, e\}$ which can love (vertical axis) and be loved (horizontal axis).
- A red box at row $x$ and column $y$ means $L(x, y) = 1$. 
1. $\forall x \exists y L(y, x)$
   Everyone is loved by someone

2. $\forall x \exists y L(x, y)$
   Everyone loves someone
3. \( \exists x \forall y L(x, y) \)
Someone loves everyone

4. \( \exists x \forall y L(y, x) \)
Someone is loved by everyone
5. $\exists x L(x, x)$
Someone loves him/herself

6. $\forall x L(x, x)$
Everyone loves him/herself
7. $\exists x \exists y L(x, y)$
   Someone loves someone
8. $\exists x \exists y L(y, x)$
   Someone is loved by someone

9. $\forall x \forall y L(x, y)$
   Everyone loves everyone
10. $\forall x \forall y L(y, x)$
    Everyone is loved by everyone
1. $\forall x \exists y L(x,y)$: Everyone is loved by someone.

2. $\forall x \exists y L(x,y)$: Everyone loves someone.

5. $\exists x L(x,x)$: Someone loves him/herself.

6. $\forall x L(x,x)$: Everyone loves him/herself.

7. $\exists x \exists y L(x,y)$: Someone loves someone.

8. $\exists x \exists y L(y,x)$: Someone is loved by someone.

9. $\forall x \forall y L(x,y)$: Everyone loves everyone.

10. $\forall x \forall y L(y,x)$: Everyone is loved by everyone.

One row/column is full:

3. $\exists x \forall y L(x,y)$: Someone loves everyone.

4. $\exists x \forall y L(y,x)$: Someone is loved by everyone.
Each interpretation, represented by a logical matrix, satisfies the formulas in its caption in a “minimal” way, i.e. whitening any red cell in any matrix would make it not satisfy the corresponding formula.

What other interpretations (matrices) satisfy formula 1? Answer: Interpretations (matrices) at 3, 6, and 10 (but not those at 2, 4, 5, and 7).

What formulas are satisfied by the interpretation (matrix) at 6? Answer: formulas 1, 2, 5, 6, 7, and 8 (but not 3, 4, 9, and 10).

Some formulas “imply” others, e.g. formula 3 “implies” formula 1, that is, each interpretation satisfying formula 3 also satisfies formula 1, but not vice versa (more later)

For this domain, which are the formulas for whom the interpretation that makes them true is unique? Answer: formulas 6, 9 and 10
Hasse diagram of implications
We now define the value of a term and of a formula in $\mathcal{L}^{\text{pred}}$, under a given interpretation $I$.

**Definition (Value of a term).**

The **value** of a term under interpretation $I$ over domain $D$ is defined by recursion:

1. If $a$ is an individual constant symbol, and $u$ a free variable symbol, then $I(a), I(u) \in D$.
2. If $t_1, t_2, \ldots t_n$ are terms, then

$$I(f(t_1, t_2, \ldots t_n)) = I(f)(I(t_1), I(t_2), \ldots, I(t_n)).$$
Value of a formula

Let \( I[\alpha = u] \) denote an interpretation over a domain \( D \) that is exactly the same as \( I \) except that \( I[\alpha = u](u) = \alpha \).

Definition (Value of a formula)

The value of formulas under interpretation \( I \) over domain \( D \) is defined by recursion:

\[
I(F(t_1, \ldots, t_n)) = \begin{cases} 
1 & \text{if } (I(t_1), I(t_2), \ldots, I(t_n)) \in I(F) \\
0 & \text{otherwise}
\end{cases}
\]

\[
I(t_1 \approx t_2) = \begin{cases} 
1 & \text{if } I(t_1) = I(t_2), \\
0 & \text{otherwise}
\end{cases}
\]

where \( F \) is a predicate symbol, and \( t_1, t_2, \ldots, t_n \) are terms.
Values of a formula

If $A$ and $B$ are formulas, then

(2)

$$I(\neg A) = \begin{cases} 1 & \text{if } I(A) = 0, \\ 0 & \text{otherwise}. \end{cases}$$

(3)

$$I(A \land B) = \begin{cases} 1 & \text{if } I(A) = I(B) = 1, \\ 0 & \text{otherwise}. \end{cases}$$

(4)

$$I(A \lor B) = \begin{cases} 1 & \text{if } I(A) = 1 \text{ or } I(B) = 1 \\ 0 & \text{otherwise}. \end{cases}$$

(5)

$$I(A \rightarrow B) = \begin{cases} 1 & \text{if } I(A) = 0 \text{ or } I(B) = 1, \\ 0 & \text{otherwise}. \end{cases}$$
Value of a formula

(6) \[ I(A \leftrightarrow B) = \begin{cases} 1 & \text{if } I(A) = I(B) \\ 0 & \text{otherwise.} \end{cases} \]

(7) \[ I(\forall x A(x)) = \begin{cases} 1 & \text{if, constructing } A(u) \text{ from } A(x) \\ & \text{(taking } u \text{ not occurring in } A(x)), \\ & \text{for every } \alpha \in D \text{ we have } I[\alpha = u](A(u)) = 1 \\ 0 & \text{otherwise.} \end{cases} \]

(8) \[ I(\exists x A(x)) = \begin{cases} 1 & \text{if, constructing } A(u) \text{ from } A(x) \\ & \text{(taking } u \text{ not occurring in } A(x)), \\ & \text{there exists } \alpha \in D \text{ such that } I[\alpha = u](A(u)) = 1 \\ 0 & \text{otherwise.} \end{cases} \]
Theorem Suppose $I$ is an interpretation over a domain $D$, $t$ is a term, and $A \in \text{Form}(\mathcal{L}^{\text{pred}})$. Then

(1) $I(t) \in D$.
(2) $I(A) \in \{0, 1\}$.

Proof: By induction on the complexity of $t$ and $A$. 


Example

Suppose
(1) \( t = f(g(a), f(b, c)) \),
(2) \( t_1 = f(g(u), f(v, c)) \)
(3) \( A = f(g(b), g(u)) \approx g(v) \),
(4) \( B = \forall x \exists y (F(y) \land G(x, y)) \),
(5) \( C = \forall x [H(x) \land G(b, x) \rightarrow \exists y \exists z (F(y) \land F(z) \land x \approx f(y, z))] \)

Suppose \( I \) is an interpretation over domain \( \mathbb{N} \) (the set of natural numbers) such that:
\( I(a) = 1, I(b) = 2, I(c) = 3, I(u) = 4, I(v) = 5 \)
\( I(F)(x) \) means \( x \) is prime
\( I(G)(x, y) \) means \( x < y \)
\( I(H)(x) \) means \( x \) is even
\( I(f)(x, y) = x + y \)
\( I(g)(x) = x^2 \).
Example contd.

Then we have

(1) \( I(t) = 1^2 + (2 + 3) = 6 \)
(2) \( I(t_1) = 4^2 + (5 + 3) = 24 \).
(3) \( I(A) \) is \( 2^2 + 4^2 = 5^2 \) which is false,
(4) \( I(B) \) is Every natural number is less than some prime number, or There are infinitely many prime numbers which is true (Euclid’s Theorem).
(5) \( I(C) \) is Every even number greater than 2 equals the sum of two primes, the value of which has not yet been decided (Goldbach’s conjecture).

It should be noticed that interpretations for predicate logic formulas are analogous to, but not the same as, value assignments defined for propositional logic formulas.
Example

(a) Find the value of the formula

\[ A = \forall x (N(x) \rightarrow \exists y (N(y) \land G(y, x))) \]

under the interpretation \( I \) over the domain \( D = \mathbb{R} \) where

\( I(N)(x) \) means \( x \) is an integer.
\( I(G)(y, x) \) means \( y > x \)

(b) Find the value of the same formula under the interpretation \( I' \)
over the domain \( D' = \mathbb{N} \) where

\( I'(N)(x) \) means \( x \) is odd,
\( I'(G)(y, x) \) means \( y \) divides \( x \) and \( 1 < y < x \).
Satisfiability in predicate logic

Suppose $\Sigma$ is a set of formulas in $\mathcal{L}^{\text{pred}}$. We define

$$I(\Sigma) = \begin{cases} 1 & \text{if for every } B \in \Sigma, I(B) = 1, \\ 0 & \text{otherwise} \end{cases}$$

- $\Sigma \subseteq \text{Form}(\mathcal{L}^{\text{pred}})$ is satisfiable iff there is some interpretation $I$ such that $I(\Sigma) = 1$.
- When $\Sigma \subseteq \text{Form}(\mathcal{L}^{\text{pred}})$ and $I(\Sigma) = 1$ we say that $I$ satisfies $\Sigma$ (or $I$ is a model of $\Sigma$), or $\Sigma$ is true in $I$. 
Validity of formulas in predicate logic

- A formula $A \in \text{Form}(L^{\text{pred}})$ is valid iff for every interpretation $I$, $I(A) = 1$.
- Validity is also called universal validity.
• A valid formula is one that is true on account of its form alone, irrespective of the meaning of the non-logical symbols (parameters) and the free variable symbols yielded under interpretations and value assignments.

• Validity is intended to capture the informal notion of truth of propositions with attention to the logical form, and in abstraction from the matter.

• A satisfiable formula (or set of formulas) is one that is true relative to some particular interpretation and value assignment. Hence, satisfiability corresponds to the informal notion of truth of propositions which follows from the matter.
Example

Suppose $A = f(g(a), g(u)) \approx g(b)$, $I$ is an interpretation over the domain $D = \mathbb{N}$ such that $I(a) = 3$, $I(b) = 5$, $I(u) = 4$, $I(f)(x, y) = x + y$, and $I(g)(x) = x^2$.

Then $I(A)$ is the true proposition

$$3^2 + 4^2 = 5^2.$$

Hence, $A$ is satisfiable. The truth of (1) is determined by the matter. In fact, there are other interpretations and assignments which make $A$ true.

$A$ is not valid. If we set $I(b) = 6$ in the above interpretation, $I(A)$ will be false.
Example

Suppose now that $B = F(u) \lor \neg F(u)$, $I$ is any interpretation.
Then, $I(B)$ is the true proposition

$$ (2) \quad I(u) \text{ has, or has not, the property } I(F). $$

The truth of (2) is not concerned with the domain, the individual $I(u)$ or the property $I(F)$. It follows from the logical form which justifies the validity of $B$. 
Remarks

- Valid formulas in $L^{\text{pred}}$ are the counterpart of tautologies in $L^p$.
- The similarities between them are obvious, but there is one important difference.
- To decide whether or not a formula of $L^p$ is a tautology, algorithms are used (for instance the truth table method).
- However, in order to know whether a formula of $L^{\text{pred}}$ is valid, we have to consider all interpretations over domains of different sizes.
- In case of an infinite domain, the procedure is in general not finite.
- Given an interpretation we are not provided a method for evaluating the value of $\forall x B(x)$ or $\exists x B(x)$ in a finite number of steps, because it presupposes the values of $B(u)$ for infinitely many $u$ in $D$.
- It is sometimes possible to decide for certain formulas of $\mathcal{L}^{pred}$ whether they are valid or not.
- However, in the general case we have the following result.
Theorem There is no algorithm for deciding the validity or satisfiability of formulas in $\mathcal{L}^{\text{pred}}$. (Church, 1936)

Alonzo Church (1903 -1995)
First-order vs. higher-order logic

- In **first-order logic**, the variables range over individuals from the domain. The quantifiers are interpreted in the familiar way as “for all individuals of the domain” and “there exist some individual of the domain”.

- In **second-order logic**, we allow as variables subsets of the domain and relations on the domain:
  - Each non-empty subset of natural numbers has a smallest element
  - Each bounded non-empty set of real numbers has a supremum

Here we have to take all subsets of the domain into consideration, and require variables and quantifiers for sets (not only for individuals in the domain).

- In **higher-order logic**, variables and quantifiers for sets of sets, sets of sets of sets, etc will be allowed.
Where we are now, chronologically

- **Aristotle** (384 - 322 BC) - earliest study of formal logic
- **George Boole** (1815-1864) - propositional logic
- **Gottlob Frege** (1848-1925) - predicate logic

- Euclid’s Theorem ("There are infinitely many primes") cannot be expressed in propositional logic, but can be expressed in predicate logic
- **Predicate logic** was intended to express all mathematics.
Frege’s second-order logic

- Frege did not restrict to first-order logic. He thought he put all mathematics on firm logical foundation. Just before the publication of Frege’s work, Bertrand Russell (1872-1970) pointed out a paradox at the heart of Frege’s system:

\[ R = \text{Set consisting of all sets that are not members of themselves.} \]

- Is \( R \) a member or itself?
How to avoid Russel’s paradox?

- “Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion.” (Frege)
- Restricting to first-order logic is one way to avoid Russel’s paradox.