Proving Argument Validity in Predicate Logic

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Logical consequences in $L^{pred}$, which are the counterparts of tautological consequences in $L^p$, involve semantics.

- The notation $\models$ for tautological consequences is also used for logical consequences.

**Definition.** Suppose $\Sigma$ is a set of formulas in $L^{pred}$ and $A$ is a formula in $L^{pred}$. $A$ is called a logical consequence of $\Sigma$, written as $\Sigma \models A$, iff for any interpretation $I$, if $I(\Sigma) = 1$ then $I(A) = 1$.

- The notations $\not\models$ and $\models$ are used in the same sense as in propositional logic.

- Two formulas are called logically equivalent (or equivalent for short, if no confusion will arise) iff $A \models B$ holds.
Prove that an argument in $\mathcal{L}^{\text{pred}}$ is valid

Prove that

$$\forall x \neg A(x) \models \neg \exists x A(x).$$

Proof (by contradiction):

Suppose the contrary, that is, suppose that there is some interpretation $I$ over a domain $D$ such that:

(1) $I(\forall x \neg A(x)) = 1$

(2) $I(\neg \exists x A(x)) = 0$.

By negating both sides in (2), we have

(3) $I(\exists x A(x)) = 1$.

that is,

(4) $I(A(d)) = 1$ for some $d \in D$. 
(4) implies $I(\neg A(d)) = 0$.

On the other hand, (1) states $I(\forall x \neg A(x)) = 1$. This implies that, in particular, $A(x)$ is true for $x = d$, that is,

$\text{(5) } I(\neg A(d)) = 1$.

We have reached a contradiction ((4) contradicts (5)), therefore our assumption was false.

Since our assumption (that the argument was invalid) was false, its opposite is true, that is, the argument is valid (sound, correct).

Q.E.D.
The particular case of empty premises: (universally) valid formulas in $\mathcal{L}^{\text{pred}}$

- Recall that the notion of (universally) valid formula in $\mathcal{L}^{\text{pred}}$ is analogous to the notion of tautology in propositional logic.
- For any formula $A$ in $\mathcal{L}^{\text{pred}}$, one has $\emptyset \models A$ if and only if $A$ is (universally) valid.
- To demonstrate that a formula $A$ is (universally) valid, we have to show that $\emptyset \models A$ under all possible interpretations.
- In other words, a formula $A$ is (universally) valid if there is no interpretation for which $A$ yields false.
Example

Show that the following formula is (universally) valid, that is:

$$\emptyset \models \forall x P(x) \lor \forall x Q(x) \rightarrow \forall x (P(x) \lor Q(x)).$$
To demonstrate that a formula $A$ is not (universally) valid, it is therefore sufficient to find an single counterexample: that is, it is sufficient to give a single interpretation for which $A$ yields false.

In the case when $A$ is of the form $B \rightarrow C$, $A$ is false iff $B$ is true while $C$ is false. Consequently, to prove that a conditional is not (universally) valid, it is sufficient to find an interpretation that makes the antecedent true and the consequent false.
Example

Prove that the following formula is not (universally) valid, that is

$$\emptyset \not\models \exists x P(x) \rightarrow \forall x P(x) \quad (\ast)$$

Note that if $P(x)$ is true for some $x$, this obviously does not justify the conclusion that $P(x)$ is true for all $x$. For instance, if a program runs for some input data, this obviously does not allow one to conclude that the program runs for all possible input data.

Thus, to prove that $(\ast)$ is not (universally) valid, one must find an interpretation that makes the antecedent $\exists x P(x)$ true and the consequent $\forall x P(x)$ false.
One such interpretation is the following.

Choose a domain $D = \{a, b\}$ and assign $P(a)$ the value 1 and $P(b)$ the value 0.

Then there is an $x$, that is, $x = a$ such that $P(x)$ is true, which means that $\exists x P(x)$ yields true.

However, $\forall x P(x)$ is obviously false.

This implies that $\exists x P(x) \rightarrow \forall x P(x)$ is not (universally) valid, that is,

\[
\emptyset \not\models \exists x P(x) \rightarrow \forall x P(x)
\]
Example

Prove that the following formula is not (universally) valid, that is,

$$
\emptyset \not\models \forall x (P(x) \lor Q(x)) \rightarrow \forall x P(x) \lor \forall x Q(x).
$$

Solution. We find an interpretation that uses a domain in which $a$ and $b$ are the only individuals, $D = \{a, b\}$. The interpretation is given by the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(x)$</th>
<th>$Q(x)$</th>
<th>$P(x) \lor Q(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

It is easy to see that this assignment makes the left side of the formula true and the right side false.
Another interpretation that proves that the above formula is not (universally) valid is the following. Assume the domain $D$ is the set of all people, and that if $P(x)$ and $Q(x)$ mean, respectively, $x$ has brown shoes and $x$ has black shoes. Under this interpretation,

- $\forall x(P(x) \lor Q(x))$ means that everyone has either brown shoes or black shoes,
- $\forall xP(x) \lor \forall xQ(x)$ means that either everyone has brown shoes or that everyone has black shoes.

In other words, in addition to the fact that all shoes are either brown or black, all shoes must be of the same colour in order to satisfy the right side of the formula. This is obviously a stronger condition than merely requiring that all shoes be either brown or black.
Proving that a formula is (universally) valid

- The problem of proving whether or not a formula in $\mathcal{L}^{\text{pred}}$ is (universally) valid is undecidable; that is, there is no generally applicable algorithm that, given an arbitrary formula in $\mathcal{L}^{\text{pred}}$ as input, can reliably determine whether or not the formula is (universally) valid.
- There are methods that work in many particular cases.
- For instance, all formulas that arise from tautologies, such as $\forall x P(x) \lor \neg(\forall x P(x))$, are (universally) valid.
- Other formulas are (universally) valid by definition. For instance, $\forall x P(x)$ must imply $P(t)$ for every term $t$, so the formula $\forall x P(x) \rightarrow P(a)$, where $a$ is in the domain, is (universally) valid.
- However, there is no algorithm to provide an answer (universally valid or not), in all cases.
Penguins are black and white. Some old TV shows are black and white. Therefore, some penguins are old TV shows.

Logic: another thing that penguins aren’t very good at.
Proof that penguins are not good at logic

Premise 1: $\forall x (\text{penguin}(x) \rightarrow \text{blackwhite}(x))$

Premise 2: $\exists x (\text{tvs}(x) \land \text{blackwhite}(x))$

Conclusion: $\exists x (\text{penguin}(x) \land \text{tvs}(x))$

Proof: Take the interpretation with domain $D = \{\text{Pingu, OldTVS}\}$, where Pingu is a black-and-white penguin, and OldTVS is a black-and-white TV show, while the values of the predicates $\text{penguin}(x)$, $\text{tvs}(x)$, $\text{blackwhite}(x)$ are:

<table>
<thead>
<tr>
<th></th>
<th>$\text{penguin}(x)$</th>
<th>$\text{blackandwhite}(x)$</th>
<th>$\text{tvs}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pingu</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>OldTVS</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

One can see that under this interpretation the premises are true, but the conclusion is false. Therefore, the argument is not sound.
Theorem.
Let $n \in \mathbb{N}$ and $A_1, A_2, \ldots, A_n$ be formulas in $\mathcal{L}^{pred}$. Then

$$A_1, \ldots, A_n \models A$$

if and only if

$$\emptyset \models A_1 \land \ldots \land A_n \rightarrow A.$$ 

Note. In the proof, we only care about the interpretations that make $A_1 \land A_2 \land \ldots A_n$ true, because falsehood implies anything.
• One philosopher was shocked when Bertrand Russell told him that a false proposition implies any proposition.
• The philosopher said “You mean from the statement “2+2=5” it follows that you are the Pope?”
• Russell replied “Yes.” “Can you prove it?” “Certainly.”
  • Suppose $2 + 2 = 5$.
  • Subtracting 2 from both sides we get $2 = 3$.
  • Transposing we get $3 = 2$.
  • Subtracting 1 from both sides we get $2 = 1$.
  “Now suppose the Pope and I are two. Since $2 = 1$, then the Pope and I are one. Hence I am the Pope.”
Exercises from Lewis Carroll

Lewis Carroll, (really Charles Dodgson writing under a pseudonym), the author of Alice in Wonderland is also author of several works in symbolic logic. The next exercises for proving arguments in $L^{\text{pred}}$ valid or invalid, come from his book Symbolic Logic.
Exercise 1

Consider the following argument.

a) All lions are fierce.
b) Some lions do not drink coffee.
c) Some fierce creatures do not drink coffee.

Express the argument in the language of predicate calculus, where \( L(x) \), \( F(x) \) and \( C(x) \) are the statements “\( x \) is a lion”, “\( x \) is fierce” and “\( x \) drinks coffee” respectively. The universe of discourse is the set of all creatures. Does c) follow from a) and b)?
Exercise from Lewis Carroll

Let $P(x)$, $I(x)$ and $V(x)$ be the statements “$x$ is a professor”, “$x$ is ignorant”, and “$x$ is vain”. Express each of the following statements in $\mathcal{L}^{\text{pred}}$ where the universe of discourse is the set of all people.

a) No professors are ignorant.
b) All ignorant people are vain.
c) No professors are vain.

Does c) follow from a) and b)?
Exercise from Lewis Caroll

Let $B(x)$, $L(x)$, $C(x)$ and $D(x)$ be the statements “$x$ is a baby”, “$x$ is logical”, “$x$ is able to manage crocodiles” and “$x$ is despised”. Express each of the following statements in $\mathcal{L}^{\text{pred}}$ if the universe of discourse is the set of all people:

a) Babies are illogical.
b) Nobody is despised who can manage a crocodile.
c) Illogical persons are despised.
d) Babies cannot manage crocodiles.

Does d) follow from a), b) and c)?