Formal (Natural) Deduction for Predicate Calculus

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The formal deducibility in predicate calculus (first-order logic) is analogous to that in propositional logic, except that is defined by some additional rules of formal deduction.

The 11 rules of formal deduction for propositional logic are included in first-order logic, but the formulas occurring in them are now formulas in $\mathcal{L}^{pred}$.

The additional rules concerning the quantifiers and equality symbol are as follows.
Additional rules of formal deduction

(12) $(\forall-)$ If $\Sigma \vdash \forall x A(x)$ is a theorem
    then $\Sigma \vdash A(t)$, where $t$ is any term, is a theorem.

(13) $(\forall+)$ If $\Sigma \vdash A(u)$ is a theorem and $u$ does not occur in $\Sigma$
    then $\Sigma \vdash \forall x A(x)$ is a theorem.

(14) $(\exists-)$ If $\Sigma, A(u) \vdash B$ is a theorem,
    and $u$ does not occur in $\Sigma$ or in $B$
    then $\Sigma, \exists x A(x) \vdash B$ is a theorem.

(15) $(\exists+)$ If $\Sigma \vdash A(t)$ is a theorem
    then $\Sigma \vdash \exists x A'(x)$ is a theorem,
    where $A'(x)$ results from $A(t)$ by replacing
    some (not necessarily all) occurrences of $t$ by $x$. 
Additional rules of formal deduction contd.

(16) \((\approx -)\) If \(\Sigma \vdash A(t_1)\) is a theorem and \(\Sigma \vdash t_1 \approx t_2\) is a theorem, then \(\Sigma \vdash A'(t_2)\) is a theorem, where \(A'(t_2)\) results from \(A(t_1)\) by replacing some (not necessarily all) occurrences of \(t_1\) by \(t_2\).

(17) \((\approx +)\) \(\emptyset \vdash u \approx u\) is a theorem.
Remarks

- In $(\forall-)$, the formula $A(t)$ results from $A(x)$ by substituting $t$ for all occurrences of $x$. It is the same for the cases of $(\forall+)$ and $(\exists-)$. 
- In $(\exists+)$, and $(\approx-)$ however, another kind of replacement is employed, which should be distinguished from substitution.
- The conditions $u$ not occurring in $\Sigma$ in $(\forall+)$, and $u$ not occurring in $\Sigma$ or $B$ in $(\exists-)$ call for some explanation.
The rule $(\forall +)$ means intuitively that from “Any member $\alpha$ of a set has a certain property” we can deduce that “Every member of the set has this property”.

For instance, in order to demonstrate that every point on the perpendicular bisector of a segment $AB$ is equidistant from $A$ and $B$, it is sufficient to prove this statement for any point $\alpha$ on the bisector.

The arbitrariness of $\alpha$ means that the choice of $\alpha$ is independent of the premises in the deduction.

This point is expressed in $(\forall +)$ by $u$ not occurring in $\Sigma$, where $u$ expresses $\alpha$, and $\Sigma$ expresses the premises.

This is similar for the case of $(\exists -)$. 
What NOT to do!

The following sequences do NOT form (formal) proofs because the rules \((\forall+)\) and \((\exists-)\) are used incorrectly in them.

Example 1.

(1) \[ A(u) \vdash A(u) \] by (Ref)
(2) \[ A(u) \vdash \forall x A(x) \] by \((\forall+)\), (1)

Example 2.

(1) \[ A(u) \vdash A(u) \] by (Ref)
(2) \[ \exists x A(x) \vdash A(u) \] by \((\exists-)\), (1)
**Formal deducibility: definition**

**Definition.** Suppose $\Sigma$ is a set of formulas in $\mathcal{L}^{\text{pred}}$ and $A$ is a formula in $\mathcal{L}^{\text{pred}}$. We say that $A$ is **formally deducible** from $\Sigma$ in first-order logic iff

$$ \Sigma \vdash A $$

can be generated by the **17 rules** of formal deduction.

- Formal deduction using the 17 rules is also called **natural deduction** for predicate (first-order) logic.
- The additional rules of formal deduction for predicate logic are similar to those in the [H&R] text, but with a different notation: $(\approx +)$ corresponds to "$\equiv i$", $(\approx -)$ to "$\equiv e$", $(\forall -)$ to "$\forall x e$", $(\forall +)$ to "$\forall x i$", $(\exists +)$ to "$\exists x i$" and $(\exists -)$ to "$\exists x e$".
Example

Prove that

\[ \neg \forall x A(x) \vdash \exists x \neg A(x). \]

Exercise: Prove that “If \( A \vdash B \) then \( \neg B \vdash \neg A \).” (\(*\)
Example continued

\[ \neg \forall x A(x) \vdash \exists x \neg A(x). \]

We now consider the converse.

1. \( \forall x A(x) \vdash \forall x A(x) \) by Ref
2. \( \forall x A(x) \vdash A(u) \) \( u \) not in \( A(x) \), 1, \((\forall -)\)
3. \( \neg A(u) \vdash \neg \forall x A(x) \) 2, \((*)\)
4. \( \exists x \neg A(x) \vdash \neg \forall x A(x) \) 3, \((\exists -)\), \( u \) does not occur elsewhere
Lemma. Suppose $A \models A'$, $B \models B'$ and $C(u) \models C'(u)$. Then:

1. $\neg A \models \neg A'$
2. $A \land B \models A' \land B'$
3. $A \lor B \models A' \lor B'$
4. $A \rightarrow B \models A' \rightarrow B'$
5. $A \leftrightarrow B \models A' \leftrightarrow B'$
6. $\forall x C(x) \models \forall x C'(x)$
7. $\exists x C(x) \models \exists x C'(x)$
Theorem. (Replacement of equivalent formulas) Let $A, B, C \in \text{Form}(\mathcal{L}^{\text{pred}})$ with $B \vdash \vdash C$. Let $A'$ result from $A$ by substituting some (not necessarily all) occurrences of $B$ by $C$. Then $A' \vdash \vdash A$.

Theorem. (Complementation) Suppose $A$ is a formula composed of atoms of $\mathcal{L}^{\text{pred}}$, the connectives $\neg$, $\lor$, $\land$ and the two quantifiers by the formation rules concerned, and $A'$ is the formula obtained by exchanging $\lor$ and $\land$, $\exists$ and $\forall$, and negating all atoms. Then $A' \vdash \vdash \neg A$. 
Soundness and completeness of formal deduction (17 rules) for predicate logic

Theorem. (Soundness and Completeness)

Let \( \Sigma \subseteq \text{Form}(\mathcal{L}^{\text{pred}}) \) and \( A \in \text{Form}(\mathcal{L}^{\text{pred}}) \). Then \( \Sigma \vdash A \) if and only if \( \Sigma \models A \).

The theorem states that the formal natural deduction system for first-order logic is sound (\( \Sigma \vdash A \) implies \( \Sigma \models A \)) and complete (\( \Sigma \models A \) implies \( \Sigma \vdash A \)). For its interpretation see the comments following its counterpart for propositional logic.
From $\forall x P(x)$ we should be able to derive $P(t)$ for any term $t$.

For instance, if $P(x)$ stands for $x$ is sleeping, then $\forall x P(x)$ means Everyone is sleeping, and from this we should be able to derive, say, that Michael is sleeping. More formally,

$(\forall -) : \text{If } \Sigma \vdash \forall x A(x), \text{ then } \Sigma \vdash A(t) \text{ for any term } t$.

Note: $t$ can be an individual, a free variable or a function of some other terms. Beware of variable clashes.
Example of using $(\forall \neg \neg)$

From the premises

All human beings are mortal.
Socrates is a human being.

prove that

Socrates is mortal.

Let $H(x)$ indicate that $x$ is human and $M(x)$ that $x$ is mortal. Furthermore, let $S$ stand for Socrates. Then, we have to prove that:

$$\forall x (H(x) \rightarrow M(x)), H(S) \vdash M(S).$$
Denote $\Sigma = \{\forall x (H(x) \rightarrow M(x)), H(S)\}$. Then,

1. $\Sigma \vdash \forall x (H(x) \rightarrow M(x))$ ($\in$)
2. $\Sigma \vdash H(S) \rightarrow M(S)$ 1, (\forall-)
3. $\Sigma \vdash H(S)$ ($\in$)
4. $\Sigma \vdash M(S)$ 2, 3, (\rightarrow -)
Semantic interpretation:

1. All humans are mortal.
2. If Socrates is human then he is mortal.
3. Socrates is human.
4. Therefore, Socrates is mortal.
Comments of Universal Generalization ($\forall+$)

($\forall+$) If $\Sigma \vdash A(u)$, $u$ not occurring in $\Sigma$ as a free variable, then $\Sigma \vdash \forall x A(x)$.

In other words, if $u$ does not appear as a free variable in any premise, one can generalize over $u$. 
• If $u$ would appear free in any premise, then $u$ would always refer to the same individual, and is fixed in this sense. For example, if $P(u)$ appears in a premise, then $P(u)$ is only true for $u$ and not necessarily true for any other individual.

• If $u$ is fixed, one cannot generalize over $u$. Generalizations from one particular individual towards the entire population are unsound.

• If, on the other hand, $u$ does not appear in any premise as a free variable, then $u$ is assumed to stand for anyone, and universal generalization may be applied without restriction.
Example of using \((\forall +)\)

Consider a problem whose domain is the set of all students in this class. Assume that all students in this class are computer science students, and that computer science students like programming.

Prove by formal deduction that all students in this class like programming.

If \(P(x)\) and \(Q(x)\) stand for \(x\) is a computer science student and \(x\) likes programming, respectively, the premises become:

\[
\forall x P(x), \quad \forall x (P(x) \rightarrow Q(x)).
\]

The desired conclusion is \(\forall x Q(x)\).
If $\Sigma = \{ \forall x P(x), \ \forall x (P(x) \rightarrow Q(x)) \}$ then

1. $\Sigma \vdash \forall x P(x)$  
   \hspace*{1cm} (\epsilon)

2. $\Sigma \vdash \forall x (P(x) \rightarrow Q(x))$  
   \hspace*{1cm} (\epsilon)

3. $\Sigma \vdash P(u)$  
   \hspace*{1cm} 1, (\forall-)

4. $\Sigma \vdash P(u) \rightarrow Q(u)$  
   \hspace*{1cm} 2, (\forall-)

5. $\Sigma \vdash Q(u)$  
   \hspace*{1cm} 3, 4, (\rightarrow -)

6. $\Sigma \vdash \forall x Q(x)$  
   \hspace*{1cm} 5, (\forall+), u does not occur in $\Sigma$
1. Everyone student in this class is a CS student.
2. CS students like programming.
3. $u$ is a CS student.
4. If $u$ is a CS student then he/she likes programming.
5. $u$ likes programming
6. Every CS student in this class likes programming.

The generalization in line 6 is possible only because $u$ does not appear in any of the premises as a free variable.
Comments on Existential Generalization

(∃+)

If Aunt Cordelia is 100 years old, then there is obviously someone, that is, Aunt Cordelia, who is over 100.

If there is any term \( t \) for which \( P(t) \) holds, then one can conclude that some \( x \) satisfies \( P(x) \). Formally,

\[
(∃+) \quad \text{If} \ \Sigma \vdash A(t) \ \text{then} \ \Sigma \vdash ∃xA(x) \ \text{where} \ A(x) \ \text{results by replacing some (not necessarily all) occurrences of} \ t \ \text{by} \ x.
\]

This rule of inference is sometimes called existential generalization.
The following example demonstrates how to use existential generalization within a formal proof.

The premises of our derivation are
1. Everybody who has won a million is rich.
2. Mary has won a million.
We want to show that the two statements formally imply that
3. There is somebody who is rich.

If the domain is the set of all people, and we denote by $W(x)$ the statement $x$ has won a million, $R(x)$ that $x$ is rich and $M$ stands for Mary, then we have to prove that

$$\forall x(W(x) \rightarrow R(x)), \quad W(M) \vdash \exists x R(x).$$
Let $\Sigma = \{\forall x (W(x) \rightarrow R(x)), \ W(M)\}$.

1. $\Sigma \vdash \forall x (W(x) \rightarrow R(x))$ (\(\in\))
2. $\Sigma \vdash W(M) \rightarrow R(M)$ 1, (\(\forall\)–)
3. $\Sigma \vdash W(M)$ (\(\in\))
4. $\Sigma \vdash R(M)$ 2, 3, (\(\rightarrow\) –)
5. $\Sigma \vdash \exists x R(x)$ 4, (\(\exists\) +)
1. Everybody who has won a million is rich.
2. Hence, if Mary has won a million, she is rich.
3. Mary has won a million.
4. Consequently, Mary is rich.
5. Somebody (Mary) is rich.
Comments on Existential Instantiation ($\exists -$)

$(\exists -$) If $\Sigma, A(u) \vdash B$, $u$ not occurring in $\Sigma$ or $B$ as a free variable, then $\Sigma, \exists x A(x) \vdash B$.

Suppose there is someone who has won a million dollars, and we want to prove that there is someone who is rich. Hence, the premises are

1. Someone has won a million dollars.
2. Everybody who has won a million dollars is rich.

We want to show that these statements imply

3. There is someone who is rich.

If the domain is the set of all people and we use the notations from the previous example, we have to prove that

$$\exists x W(x), \forall x (W(x) \rightarrow R(x)) \vdash \exists x R(x).$$
Denote by $\Sigma = \{\forall x(W(x) \rightarrow R(x))\}$.

1. $\Sigma, W(u) \vdash \forall x(W(x) \rightarrow R(x))$ (\(\in\))
2. $\Sigma, W(u) \vdash W(u) \rightarrow R(u)$ 1, (\(\forall-\))
3. $\Sigma, W(u) \vdash W(u)$ (\(\in\))
4. $\Sigma, W(u) \vdash R(u)$ 2, 3, (\(\rightarrow-\))
5. $\Sigma, W(u) \vdash \exists x R(x)$ 4, (\(\exists+\))
6. $\Sigma, \exists x W(x) \vdash \exists x R(x)$ 5, (\(\exists-\))

\(u\) does not occur elsewhere
Semantic interpretation:

We first change our premises to Everybody who has won a million is rich, and \( u \) has won a million dollars.

Then,

1. Everybody who has won a million is rich.
2. If \( u \) has won a million then he is rich.
3. \( u \) has won a million.
4. Therefore, \( u \) is rich.
5. Under these assumptions (that \( u \) has won a million), there is somebody who is rich.
6. As someone has won a million dollars, our assumptions are valid, therefore there is somebody who is rich.
(∃—) is called existential instantiation because, instead of the premise
There is somebody who has won a million dollars
we could use the instance \( u \) has won a million dollars,
even though we did not know who the particular \( u \) that made
\( \exists x W(x) \) true was.

Then we could carry on with the proof, and replace in the end \( W(u) \)
by \( \exists x W(x) \).
Special attention has to be paid to the following condition:

For $W(u)$ to be replaced by $\exists x W(x)$, *$u$ must appear neither in the premises nor in the conclusion.*

In particular, Steps 5, 6 could not be interchanged.
How to apply \((\rightarrow +)\) in predicate calculus

Let \(S(x)\) stand for \(x\) studied and \(P(x)\) stand for \(x\) passed. The premise is that everyone who studied passed. Prove that everyone who did not pass did not study.

Translation:

\[
\forall x (S(x) \rightarrow P(x)) \vdash \forall x [\neg P(x) \rightarrow \neg S(x)]
\]
Denote $\Sigma = \{\forall x(S(x) \rightarrow P(x))\}$.

1. $\Sigma \vdash \forall x(S(x) \rightarrow P(x))$ (\$\in\$)
2. $\Sigma \vdash S(u) \rightarrow P(u)$ 1, ($\forall -$)
3. $\Sigma \vdash \neg P(u) \rightarrow \neg S(u)$ 2, contrapositive
4. $\Sigma \vdash \forall x(\neg P(x) \rightarrow \neg S(x))$ 3, ($\forall +$), $u$ not in $\Sigma$
1. Everyone who studied passed.

2. If $u$ studied, he passed.

3. If $u$ did not pass then he cannot have studied.

4. Everyone who did not pass cannot have studied.

Note that the generalization in line 7 is only possible because $u$ does not occur free in any premise.
1. Prove by formal deduction that Paul is the son of John from the following premises:

John is the father of Paul.
Paul is not the daughter of John.
A child is either a son or a daughter.

Use the following predicates:

\( F(x, y) : x \) is the father of \( y \).
\( S(x, y) : x \) is the son of \( y \).
\( D(x, y) : x \) is the daughter of \( y \).

Use \( J \) for John and \( P \) for Paul. The domain is the set of all people.

**Hint:** the formal proof uses the disjunctive syllogism 
\((A \lor B, \neg A \vdash B)\) and \((\forall \neg)\), universal instantiation.
2. Prove by using formal deduction that:

\[ \forall x \forall y P(x, y) \vdash \forall y \forall x P(x, y). \]

(validity of interchanging universal quantifiers)

**Hint**: Use \((\forall +)\), the universal generalization.

3. Prove by using formal deduction that:

\[ \forall x P(x) \vdash \forall y P(y) \]

(variable substitution)

**Hint**: Use \((\forall +)\), the universal generalization.
4. Prove by using formal deduction that:

\[ \neg \exists x P(x) \vdash \forall x \neg P(x) \]

(negation of formulas with existential quantifier - direct implication)

**Hint:** Use the existential generalization (\(\exists+\)).

5. Prove by using formal deduction that:

\[ \forall x \neg P(x) \vdash \neg \exists x P(x) \]

(negation of formulas with existential quantifier - the converse)

**Hint:** Use existential instantiation, (\(\exists−\)).
Exercises: Proving arguments valid with semantic ($\models$) or syntactic ($\vdash$) methods

Give a semantic proof of the validity or invalidity of the following argument (using the notion of tautological consequence, $\models$). If the argument is valid, give also a proof by formal deduction, $\vdash$.

Premise 1: $\forall x (P(x) \rightarrow Q(x))$
Premise 2: $\exists x (R(x) \land \neg Q(x))$
Premise 3: $\forall x (R(x) \rightarrow P(x) \lor S(x))$

Conclusion: $\exists x (R(x) \land S(x))$
Exercise

Give a semantic proof of the validity or invalidity of the following argument (using the notion of tautological consequence, \( \models \)). If the argument is valid, give also a proof by formal deduction, \( \vdash \).

Premise 1: \( \forall x \forall y (D(x) \land E(y) \rightarrow F(x, y)) \)
Premise 2: \( \forall x \forall y (D(x) \land F(x, y) \rightarrow G(y)) \)

Conclusion: \( \exists x D(x) \rightarrow \forall x (E(x) \rightarrow G(x)) \)
Translate the following argument in the language of predicate calculus and determine whether it is valid or invalid. The universe of discourse is the set of all birds. (From Lewis Caroll.)

1) All hummingbirds are richly coloured.
2) No large birds live one honey.
3) Birds that do not live on honey are dull in colour.

Hummingbirds are small.

Use the following predicate symbols: $H(x)$ - “$x$ is a hummingbird”, $L(x)$ - “$x$ is large”, $H(x)$ - “$x$ lives on honey”, $R(x)$ - “$x$ is richly coloured”.

![Hummingbird image]
Translate the following argument in the language of predicate calculus and determine whether it is valid or invalid. (From Lewis Caroll.)

Only ghosts live in the haunted house.
No happy ghost is intelligent.
No ghost is happy.

Anyone who lives in the haunted house is either intelligent or unhappy.

Use the following predicate symbols: $G(x)$ - “$x$ is a ghost”, $L(x)$ - “$x$ lives in a haunted house”, $H(x)$ - “$x$ is happy”, $I(x)$ - “$x$ is intelligent.”. The universe of discourse consists of people and ghosts.
Give a semantic proof of the validity or invalidity of the following argument (using the notion of tautological consequence, $|=\ )$. If the argument is valid, give also a proof by formal deduction, $\vdash$.

Premise (1) $\exists x M(x)$
Premise (2) $\forall x (M(x) \rightarrow \exists y C(x, y))$
Premise (3) $\forall x (\exists y C(x, y) \rightarrow F(x))$

Conclusion: $\exists y F(y)$