Propositional Calculus:
Formula Simplification, Essential Laws, Normal Forms

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Propositional calculus

In standard algebra, expressions in which the variables and constants represent numbers are manipulated. Consider for instance the expression

\[(a + b) - b\]

One sees at a glance that this expression yields \(a\). In fact, we are so accustomed to performing such algebraic manipulations that we are no longer aware of what is behind each step. Here we used the identities

\[(x + y) - z = x + (y - z)\]

\[y - y = 0\]

\[x + 0 = x\]
Simplifications of logic formulas

Consider the following formula

\[(p \land q) \land \neg q.\]

This formula can be simplified in a similar way, except that (tauto)logical equivalences take place of algebraic identities.

\[(A \land B) \land C \equiv A \land (B \land C)\]

\[(A \land \neg A) \equiv 0\]

\[A \land 0 \equiv 0\]

We can now apply these equivalences to conclude

\[(p \land q) \land \neg q \equiv p \land (q \land \neg q) \equiv p \land 0 \equiv 0.\]
Removing conditionals and biconditionals

Since the symbolic treatment of conditionals and biconditionals is relatively cumbersome, one usually removes them before performing further formula manipulations.

To remove the conditional, one uses the following logical equivalence:

\[ p \rightarrow q \equiv \neg p \lor q. \]

There are two ways to remove the biconditional

\[ p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q) \]

\[ p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p) \equiv (\neg p \lor q) \land (p \lor \neg q). \]

The first version expresses the fact that two formulas are equivalent if they have the same truth values.

The second version stresses the fact that two formulas are equivalent if the first implies the second and the second implies the first.
Example. Remove $\rightarrow$ and $\leftrightarrow$ from the following formula:

$$(p \rightarrow q \land r) \lor ((r \leftrightarrow s) \land (q \lor s)).$$

Solution.

$$(\neg p \lor q \land r) \lor (((\neg r \lor s) \land (r \lor \neg s)) \land (q \lor s)).$$
## Essential laws for propositional calculus

<table>
<thead>
<tr>
<th>Law</th>
<th>Name</th>
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<tbody>
<tr>
<td>( p \lor \neg p \models 1 )</td>
<td>Excluded middle law</td>
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<tr>
<td>( p \land \neg p \models 0 )</td>
<td>Contradiction law</td>
</tr>
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<td>( p \lor 0 \models p, p \land 1 \models p )</td>
<td>Identity laws</td>
</tr>
<tr>
<td>( p \lor 1 \models 1, p \land 0 \models 0 )</td>
<td>Domination laws</td>
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<td>( p \lor p \models p, p \land p \models p )</td>
<td>Idempotent laws</td>
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<td>( \neg(\neg p) \models p )</td>
<td>Double-negation law</td>
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<td>( p \lor q \models q \lor p, p \land q \models q \land p )</td>
<td>Commutative laws</td>
</tr>
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<td>( (p \lor q) \lor r \models p \lor (q \lor r) )</td>
<td>Associative laws</td>
</tr>
<tr>
<td>( (p \land q) \land r \models p \land (q \land r) )</td>
<td>Associative laws</td>
</tr>
<tr>
<td>( (p \lor q) \land (p \lor r) \models p \lor (q \land r) )</td>
<td>Distributive laws</td>
</tr>
<tr>
<td>( (p \land q) \lor (p \land r) \models p \land (q \lor r) )</td>
<td>Distributive laws</td>
</tr>
<tr>
<td>( \neg(p \land q) \models \neg p \lor \neg q )</td>
<td>De Morgan’s laws</td>
</tr>
<tr>
<td>( \neg(p \lor q) \models \neg p \land \neg q )</td>
<td>De Morgan’s laws</td>
</tr>
</tbody>
</table>
Essential laws for propositional calculus

- All laws can be proved by the truth table method.
- With the exception of the double-negation law all laws come in pairs, called dual pairs. For each formula depending only on the connectives $\neg$, $\land$, $\lor$, the dual is found by replacing all 1 by 0 and all 0 by 1 and by replacing all $\land$ by $\lor$ and all $\lor$ by $\land$.
- The laws allow one to simplify a formula and it is normally a good idea to apply them whenever it is possible. For instance, the formula $\neg\neg p \land (q \lor \neg q)$ is logically equivalent to $p$.
- The commutative, associative and distributive laws have their equivalents in standard algebra. In fact, the connective $\lor$ is often treated like $+$, and the connective $\land$ is often treated like $\times$. (The analogy sometimes breaks down.)
Further laws

- From these laws one can derive further laws, for example, the absorption laws

\[ p \lor (p \land q) \models p \]
\[ p \land (p \lor q) \models p. \]

(To prove them, use identity law, distributive law, domination law and identity law again.)

- Another important law (and its dual):

\[ (p \land q) \lor (\neg p \land q) \models q \]
\[ (p \lor q) \land (\neg p \lor q) \models q. \]
Definition. A formula is called a literal if it is of the form $p$ or $\neg p$, where $p$ is a propositional variable. The two formulas $p$ and $\neg p$ are called complementary literals.

The following rules are available to simplify conjunctions containing only literals.

- If a conjunction contains complementary literals or if it contains the propositional constant 0, it always yields 0; that is, it is a contradiction.
- All instances of the propositional constant 1 and all duplicate copies of any literal, may be dropped.
Shortcuts for simplifying formulas

To simplify disjunctions, the duals of the previous two rules are used.

- If a disjunction contains complementary literals or if it contains the propositional constant 1, it always yields 1; that is, it is a tautology.
- All instances of the propositional constant 0 and all duplicate copies of any literal may be dropped.

Example. Simplify

\[(p_3 \land \neg p_2 \land p_3 \land \neg p_1) \lor (p_1 \land p_3 \land \neg p_1).\]

Solution: \(\neg p_1 \land \neg p_2 \land p_3.\)
Disjunctive Normal Form (DNF)

Formulas can be transformed into standard forms so that they become more convenient for symbolic manipulations and make identification and comparison of two formulas easier. There are two types of normal forms in propositional calculus: the Disjunctive Normal Form (DNF) and the Conjunctive Normal Form (CNF).

**Definition.** A formula is said to be in disjunctive normal form if it is written as a disjunction, in which all the terms are conjunctions of literals.

**Example:** $(p \land q) \lor (p \land \neg q)$, $p \lor (q \land r)$, $\neg p \lor t$ are in disjunctive normal forms. The disjunction $\neg (p \land q) \lor r$ is not in normal form. A formula in disjunctive normal form is of the form $$(A_{11} \land \ldots \land A_{1n_1}) \lor \ldots \lor (A_{k1} \land \ldots \land A_{kn_k})$$ where $A_{ij}$ are literals.
Conjunctive normal form

**Definition.** A disjunctions (conjunction) with literals as disjuncts (conjuncts) is called a **disjunctive (conjunctive) clause**. Disjunctive and conjunctive clauses are simply called clauses.

**Definition.** A conjunction with disjunctive clauses as its conjuncts is said to be in **conjunctive normal form**.

**Example:** $p \land (q \lor r)$ and $p \land q$ are in conjunctive normal form. However $p \land (r \lor (p \land q))$ is not in conjunctive normal form.

A formula in conjunctive normal form is of the form $(A_{11} \lor \ldots \lor A_{1n_1}) \land \ldots \land (A_{k1} \lor \ldots \lor A_{kn_k})$, where $A_{ij}$ are literals. The formulas $(A_{j1} \lor \ldots \lor A_{jn_j})$ are the disjunctive clauses of the formula in CNF.
Normal form examples

Observe the following formulas:

(1) $p$
(2) $\neg p \lor q$
(3) $\neg p \land q \land \neg r$
(4) $\neg p \lor (q \land \neg r)$
(5) $\neg p \land (q \lor \neg r) \land (\neg q \lor r)$
Example contd.

• (1) is an atom, and therefore a literal. It is a disjunction with only one disjunct. It is also a conjunction with only one conjunct. Hence it is a disjunctive or conjunctive clause with one literal. It is a formula in disjunctive normal form with one conjunctive clause \( p \). It is also a formula in conjunctive normal form with one disjunctive clause \( p \).

• (2) is a disjunction with two disjuncts, and a disjunctive normal form with two clauses, each with one literal. It is also a conjunction with one conjunct, and a formula in conjunctive normal form which consists of two literals.
• (3) is a conjunction and a formula in conjunctive normal form. It is also a disjunction and a formula in disjunctive normal form.
• (4) is a formula in disjunctive normal form, but not in conjunctive normal form.
• (5) is a formula in conjunctive normal form but not in disjunctive normal form.
• If $\lor$ is exchanged for $\land$ in (4) and (5), then (4) becomes a formula in conjunctive normal form and (5) a formula in disjunctive normal form.
How to obtain normal forms?

Use the following tautological equivalences:

1. \( A \rightarrow B \models \neg A \lor B \).
2. \( A \leftrightarrow B \models (\neg A \lor B) \land (A \lor \neg B) \).
3. \( A \leftrightarrow B \models (A \land B) \lor (\neg A \land \neg B) \).
4. \( \neg \neg A \models A \).
5. \( \neg (A_1 \land \ldots \land A_n) \models \neg A_1 \lor \ldots \lor \neg A_n \).
6. \( \neg (A_1 \lor \ldots \lor A_n) \models \neg A_1 \land \ldots \land \neg A_n \).
7. \( A \land (B_1 \lor \ldots \lor B_n) \models (A \land B_1) \lor \ldots \lor (A \land B_n) \).
   \( (B_1 \lor \ldots \lor B_n) \land A \models (B_1 \land A) \lor \ldots \lor (B_n \land A) \).
8. \( A \lor (B_1 \land \ldots \land B_n) \models (A \lor B_1) \land \ldots \land (A \lor B_n) \).
   \( (B_1 \land \ldots \land B_n) \lor A \models (B_1 \lor A) \land \ldots \land (B_n \lor A) \).
By the replaceability of tautological equivalences, we can replace the preceding formulas on the left with the corresponding ones on the right to yield a formula tautologically equivalent to the original one.

- By (1)–(3) we eliminate $\rightarrow$ and $\leftrightarrow$.
- By (4)–(6) we eliminate $\neg$, $\lor$, $\land$ from the scope of $\neg$ such that any $\neg$ has only an atom as its scope.
- By (7) we eliminate $\lor$ from the scope of $\land$.
- By (8) we eliminate $\land$ from the scope of $\lor$.

This method leads to obtaining the disjunctive or conjunctive normal forms.
Example. Convert the following formula into a conjunctive normal form.

\[ \neg((p \lor \neg q) \land \neg r). \]

The conjunctive normal form can be found by the following derivations:

\[ \neg((p \lor \neg q) \land \neg r) \]
\[ \equiv \neg(p \lor \neg q) \lor \neg \neg r \quad \text{De Morgan} \]
\[ \equiv \neg(p \lor \neg q) \lor r \quad \text{Double negation} \]
\[ \equiv (\neg p \land \neg \neg q) \lor r \quad \text{De Morgan} \]
\[ \equiv (\neg p \land q) \lor r \quad \text{Double negation} \]
\[ \equiv (\neg p \lor r) \land (q \lor r) \quad \text{Distributivity} \]
Algorithm for Conjunctive Normal Form

1. Eliminate equivalence and implication, using $A \rightarrow B \equiv \neg A \lor B$ and $A \leftrightarrow B \equiv (\neg A \lor B) \land (A \lor \neg B)$.

2. Push $\neg$ inwards, to the variables, using De Morgan’s Laws.

3. Recursive procedure $CNF(A)$:
   
   1. If $A$ is a literal then return $A$.
   
   2. If $A$ is $B \land C$ then return $CNF(B) \land CNF(C)$.
   
   3. If $A$ is $B \lor C$ then
      
      - call $CNF(B)$ and $CNF(C)$
      - suppose $CNF(B) = B_1 \land B_2 \ldots \land B_n$
      - suppose $CNF(C) = C_1 \land C_2 \ldots \land C_m$
      - return $\land_{i=1 \ldots n, j=1 \ldots m}(B_i \lor C_j)$
Example of Step 3.3 in converting to CNF

\[(a \lor b) \land (c \lor \neg a \lor d)) \lor ((\neg a) \land (c \lor d) \land (\neg b \lor \neg c \lor \neg d))\]

- \(n = 2\) clauses
- \(m = 3\) clauses
- the resulting CNF will have \(2 \times 3 = 6\) clauses
- it can be further simplified

\[(a \lor b \lor \neg a) \land (a \lor b \lor c \lor d) \land (a \lor b \lor \neg b \lor \neg c \lor \neg d) \land (c \lor \neg a \lor d \lor \neg a) \land (c \lor \neg a \lor d \lor c \lor d) \land (c \lor \neg a \lor d \lor \neg b \lor \neg c \lor \neg d) \]

\[(a \lor b \lor c \lor d) \land (\neg a \lor c \lor d)\]
**Existence of normal forms**

**Theorem.** Any formula $A \in \text{Form}(L^p)$ is tautologically equivalent to some formula in disjunctive normal form.

**Proof.** If $A$ is a contradiction, then $A$ is tautologically equivalent to the DNF $p \land \neg p$, $p$ being any atom occurring in $A$.

If $A$ is not a contradiction, we employ the following method.

Suppose $A$ has three atoms, $p, q, r$ occurring in $A$, and the value of $A$ is 1 iff $1, 1, 0$, or $1, 0, 1$, or $0, 0, 1$, are assigned to $p, q, r$ respectively. For each of these assignments, we form a conjunctive clause with three literals, each being one of the atoms or its negation according to whether this atom is assigned 1 or 0:

1. $p \land q \land \neg r$
2. $p \land \neg q \land r$
3. $\neg p \land \neg q \land r$
Obviously,

- (1) has value 1 iff 1, 1, 0 are assigned to $p, q, r$
- (2) has value 1 iff 1, 0, 1 are assigned to $p, q, r$
- (3) has value 1 iff 0, 0, 1 are assigned to $p, q, r$

Therefore, the following DNF is tautologically equivalent to $A$:

$$(p \land q \land \neg r) \lor (p \land \neg q \land r) \lor (\neg p \land \neg q \land r).$$

For a tautology $A$, the required DNF may simply be $p \lor \neg p$ where $p$ is any atom occurring in $A$. Q.E.D.

Similarly, we have:

**Theorem.** Any formula $A \in \text{Form}(\mathcal{L}^p)$ is tautologically equivalent to some formula in conjunctive normal form.
We have shown how to find the truth table of a logical formula. The previous theorem shows that the reverse is also possible: One can convert any given truth table into a (DNF) formula.

What is the DNF of the formula given by the following truth table?

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>1</td>
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</tbody>
</table>

\[ f \models (p \land q \land r) \lor (p \land \lnot q \land r) \lor (\lnot p \land \lnot q \land r). \]
Conjunctive normal form and complementation

- Complementation can be used to obtain conjunctive normal forms from truth tables.

- If $A$ is a formula containing only the connectives $\neg$, $\lor$ and $\land$, then its complement is formed by replacing all $\lor$ by $\land$, all $\land$ by $\lor$ and all atoms by their complements.

- Example: Find the complement of the formula $A = (p \land q) \lor \neg r$.

- Complementation can be used to find the conjunctive normal form from the truth table of some truth function (Boolean function) $f$. 
**CNF from truth tables**

One first determines the disjunctive normal form for $\neg f$. If the resulting disjunctive normal form is $A$, then $A \models \neg f$, and the complement of $A$ must be logically equivalent to $f$.

**Example:** Find the full conjunctive normal form for $f_1$ given by the table

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$f_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0</td>
<td>1</td>
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</table>
Example contd.

Solution: \( \neg f_1 \) is true for the following assignments:

\[
\begin{align*}
    p &= 1, \quad q = 0, \quad r = 1 \\
    p &= 1, \quad q = 0, \quad r = 0 \\
    p &= 0, \quad q = 0, \quad r = 1
\end{align*}
\]

The disjunctive normal form of \( \neg f_1 \) is therefore

\[
(p \land \neg q \land r) \lor (p \land \neg q \land \neg r) \lor (\neg p \land \neg q \land r).
\]

This formula has the complement

\[
f_1 \models (\neg p \lor q \lor \neg r) \land (\neg p \lor q \lor r) \land (p \lor q \lor \neg r),
\]

which is the desired conjunctive normal form.