Propositional Language - Semantics

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Syntax and semantics

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<th>Syntax</th>
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<td>(p \land \land\land))))) - syntactically incorrect</td>
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The “meaning” of a nonatomic formula, that is, its truth values (true or false) must be derived from the truth values of its constituent atomic formulas. Before this can be done, the formula must be parsed; that is, all subformulas of the formula must be found.

If you take a class in computers and if you do not understand recursion, you will not pass.

We want to know exactly when this statement is true and when it is false.

Define:

$p$: “You take a class in computers.”

$q$: “You understand recursion.”

$r$: “You pass.”

The statement becomes \((p \land \neg q) \rightarrow \neg r\).
Truth table for \((p \land \neg q) \rightarrow \neg r\)

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There are two identical twins, Jenny and Susan. One of them is a knave (always lies) and one is a knight (always tells the truth). You don’t know which one is which. You meet one of them, and you want to find out if she is Jenny or Susan by asking one yes/no question. What is the question and why does it work?

Question (for the person you are talking to): “Is Susan a knave?”

Claim: The answer to this question is “yes” iff you are talking to Jenny.
Value (truth) assignments

- A truth table list the values of a formula under all possible value assignments.
- Fix a set \( \{0, 1\} \) of truth values. We interpret 0 as false and 1 as true.
- **Definition.** A (Boolean, truth) value assignment (called also truth assignment, respectively valuation or model in the text) is a function \( \nu \)

\[
\nu : \text{Atom}(\mathcal{L}^p) \rightarrow \{0, 1\},
\]

with the set of all proposition variables as domain and \( \{0, 1\} \) as range.
Definition. The value of a formula $A$ in $\text{Form}(\mathcal{L}^p)$ with respect to the value assignment $\nu$ is defined recursively as follows:

1. If $A$ is in $\text{Atom}(\mathcal{L}^p)$ then $\nu(A)$ is in \{0, 1\} as defined previously.
2. $\nu(\neg A) = 1$ if $\nu(A) = 0$, and 0 otherwise.
3. $\nu(A \land B) = 1$ if $\nu(A) = \nu(B) = 1$, and 0 otherwise.
4. $\nu(A \lor B) = 1$ if $\nu(A) = 1$ or $\nu(B) = 1$ (or both), and 0 otherwise.
5. $\nu(A \rightarrow B) = 1$ if $\nu(A) = 0$ or $\nu(B) = 1$, and 0 otherwise.
6. $\nu(A \leftrightarrow B) = 1$ if $\nu(A) = \nu(B)$, and 0 otherwise.
Suppose $A = p \lor q \rightarrow q \land r$, and $\nu$ is a value assignment such that
\[ \nu(p) = \nu(q) = \nu(r) = 1. \]
Then we have $\nu(p \lor q) = 1$, $\nu(q \land r) = 1$ and therefore $\nu(A) = 1$.

Suppose $\nu_1$ is another value assignment,
\[ \nu_1(p) = \nu_1(q) = \nu_1(r) = 0. \]
Then we have $\nu_1(p \lor q) = 0$, $\nu_1(q \land r) = 0$ and therefore $\nu_1(A) = 1$.

If $\nu_2$ is a third value assignment such that $\nu_2(p) = 1$ and $\nu_2(r) = \nu_2(q) = 0$, then $\nu_2(A) = 0$.

The above example illustrates that the values which various value assignments assign to a formula may or may not be different.
**Satisfiability**

**Definition.** We say that a value assignment $v$ satisfies a formula $A$ in $\text{Form}(\mathcal{L}^p)$ iff $v(A) = 1$.

We use the Greek letter $\Sigma$ to denote any set of formulas.

Define

$$v(\Sigma) = \begin{cases} 1 & \text{if for each formula } B \in \Sigma, v(B) = 1, \\ 0 & \text{otherwise} \end{cases}$$

**Definition.** A set of formulas $\Sigma \subseteq \text{Form}(\mathcal{L}^p)$ is satisfiable iff there exists a value assignment $v$ such that $v(\Sigma) = 1$.

When $v(\Sigma) = 1$, then $v$ is said to satisfy $\Sigma$, and $\Sigma$ is said to be true under $v$. 

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**Propositional Language - Semantics**

CS245, Logic and Computation
Observations

- Note that $\nu(\Sigma) = 1$ means that under the value assignment $\nu$, all the formulas of $\Sigma$ are true.

- Note that $\nu(\Sigma) = 0$ means that for at least one formula $B \in \Sigma$, $\nu(B) = 0$.

  (This does not necessarily mean that $\nu(C) = 0$ for every formula $C$ in $\Sigma$.)
Sudoku as a Satisfiability Problem

In a 4 x 4 Sudoku puzzle, each value in \{1, 2, 3, 4\} must appear exactly once in each row, column, and 2x2 block.

\[
\begin{array}{ccc}
3 & 4 & \\
1 & 3 & \\
2 & 3 & \\
1 & 2 & \\
\end{array}
\]

Write a formula that requires all the following to be true:
(a) Solution must be consistent with the starting grid
(b) At most one digit per square
(c) In each row, each digit must appear exactly once
(d) In each column, each digit must appear exactly once
(e) In each block, each digit must appear exactly once
Sudoku as SAT problem

Set up variables \( v_{ijk} \), \( 1 \leq i, j, k \leq 4 \), so that \( v_{ijk} = 1 \), iff the cell at position \((i, j)\) equals number \( k \).

Solution of the Sudoku puzzle = determining the satisfiability of the formula obtained by taking the conjunction of all formulas below:

- **(a) Consistency with starting grid:** If cell \((1, 2)\) has digit 3 in it, then add the atomic formula \( v_{123} \); Do not add anything for unfilled cells.

- **(b) At most one digit per cell:** For every cell \((i, j)\), and each pair of different digits \( k, k' \), add the formula \( (v_{ijk} \rightarrow \neg v_{ijk'}) \)
(c) **In each row each digit appears exactly once:**

- **“at least once”:** Focus on row $i$ and digit $k$, add the formula
  \[(v_{i1k} \lor v_{i2k} \lor v_{i3k} \lor v_{i4k})\]
- **“at most once”:** Look at every pair of cells in a row, $(i,j)$ and $(i,j')$, $j \neq j'$, and require that they do not both contain $k$ by adding \[(v_{ijk} \rightarrow \neg v_{ij'k})\]

(d) **In each column each digit appears exactly once:** Like rows, but fixes the column $j$ and digit $k$.

(e) **In each block, each digit appears exactly once:** Same pattern as rows and columns, but the row and column indexes must vary over the cells within a given block.
Tautologies and contradictions

- **Definition** A formula $A$ is a tautology iff it is true under all possible value assignments, i.e. iff for any value assignment $v$, $v(A) = 1$.

- **Definition.** A formula $A$ is a contradiction iff it is false under all possible value assignments, i.e., iff for any value assignment $v$, $v(A) = 0$.

- **Definition** A formula that is neither a tautology nor a contradiction is called contingent.
Consider the formula $\neg(p \land q) \lor q$. Is this formula a tautology?

Truth table for a tautology

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<tr>
<th>$p$</th>
<th>$q$</th>
<th>$(p \land q)$</th>
<th>$\neg(p \land q)$</th>
<th>$\neg(p \land q) \lor q$</th>
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Law of the excluded middle ("tertium non datur") states that $p \lor \neg p$ is a tautology. In other words, $p$ is either true or false, everything else is excluded.

The table below proves this law by showing that it is true for all value assignments:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
<th>$p \lor \neg p$</th>
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Tautology: Observations

If $A$ is a tautology that contains the variable $p$, one can determine a new expression by replacing all instances of $p$ by an arbitrary formula. The resulting formula $A'$ is also a tautology.

For example, $p \lor \neg p$ is a tautology. Replace all instances of $p$ by any formula we like, say by $p \land q$. The resulting formula $A' = (p \land q) \lor \neg(p \land q)$ is again a tautology.

Theorem. Let $A$ be a tautology and let $p_1, p_2, \ldots p_n$ be the propositional variables of $A$. Suppose that $B_1, B_2, \ldots B_n$ are arbitrary formulas. Then, the formula obtained by replacing $p_1$ by $B_1$, $p_2$ by $B_2$, ..., $p_n$ by $B_n$ is a tautology.

Example. Use the fact that $\neg((p \land q) \lor q$ is a tautology to prove that $\neg((p \lor q) \land r) \lor r$ is a tautology.
Law of contradiction: “Nothing can both be, and not be”, that is, \( \neg(p \land \neg p) \) is a tautology or, equivalently, \( (p \land \neg p) \) is a contradiction.

Truth table for a contradiction

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<th>( \neg p )</th>
<th>( p \land \neg p )</th>
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Contradictions are related to tautologies: \( A \) is a tautology if and only if \( \neg A \) is a contradiction.
Three essential laws of thought - Plato

1. **Law of identity:**
   “Whatever is, is.” \( p = p \)

2. **Law of contradiction:**
   “Nothing can both be and not be.” \( \neg(p \land \neg p) \)

3. **Law of excluded middle:**
   “Everything must either be, or not be.” \( (p \lor \neg p) \)

*Plato (428 - 348 B.C.*)
Fuzzy Logic

Truth values are real numbers in the interval \([0, 1]\)
“true” = 1, “false” = 0, “partially true” = a number between 0 and 1

- \(\text{AND}(x, y) = \min\{x, y\}\)
- \(\text{OR}(x, y) = \max\{x, y\}\)
- \(\text{NOT}(x) = 1 - x\)

If the values of the variables are 0 and 1, these definitions amount to classical logic connectives

Does the law of excluded middle hold?
For \(p = 0.2\), we have \(p \lor \neg p = \max\{0.2, 0.8\} = 0.8 \neq 1\).

Does the law of contradiction hold?
For \(p = 0.2\), we have \(p \land \neg p = \min\{0.2, 0.8\} = 0.2 \neq 0\).
Proving arguments valid

Logical argument:

Premise 1
Premise 2
...
Premise \( n \)

Conclusion

Logical arguments can be

- **Correct (valid, sound)**
- **Incorrect (invalid, unsound)**
Formalizing argument validity: Tautological consequence

**Definition** Suppose \( \Sigma \subseteq \text{Form}(\mathcal{L}^p) \) and \( A \in \text{Form}(\mathcal{L}^p) \). \( A \) is a tautological consequence of \( \Sigma \) (that is, of the formulas in \( \Sigma \)), written as \( \Sigma \models A \), iff for any value assignment \( v \), we have that \( v(\Sigma) = 1 \) implies \( v(A) = 1 \).

**Observations**

- \( \models \) is not a symbol of the formal propositional language and \( \Sigma \models A \) is not a formula.
- \( \Sigma \models A \) is a statement (in the metalanguage) about \( \Sigma \) and \( A \).
- We write \( \Sigma \not\models A \) for “not \( \Sigma \models A \)”.
- \( \models \) is called “semantic entailment” in [Huth&Ryan]
The special case $\emptyset \models A$

When $\Sigma$ is the empty set, we obtain the important special case of tautological consequence, $\emptyset \models A$

By definition, $\emptyset \models A$ means

“For any value assignment $v$, if $v(\emptyset) = 1$ then $v(A) = 1$.” (*)

where $v(\emptyset) = 1$ means “For any $B$, if $B \in \emptyset$ then $v(B) = 1$” (**) 

Because $B \in \emptyset$ is false, (**) is vacuously true.

Thus, “if $v(\emptyset) = 1$ then $v(A) = 1$” in (*) is equivalent to $v(A) = 1$.

Consequently, $\emptyset \models A$ means that $A$ is a tautology.

*Intuitively speaking, $\Sigma \models A$ means that the truth of the formulas in $\Sigma$ is the sufficient condition of the truth of $A$. Since $\emptyset$ consists of no formula, $\emptyset \models A$ means that the truth of $A$ is unconditional, hence $A$ is a tautology.*
Tautological consequence and valid arguments

Let $\Sigma = \{A_1, A_2, \ldots, A_n\}$ be a set of formulas (premises) and $A$ be a formula (conclusion). The following are equivalent:

- The argument with premises $A_1, A_2, \ldots, A_n$ and conclusion $A$ is valid.
- $(A_1 \land A_2 \land \ldots \land A_n) \rightarrow A$ is a tautology.
- $(A_1 \land A_2 \ldots \land A_n \land \lnot A)$ is a contradiction.
- $A$ is a tautological consequence of $\Sigma$, i.e.

$$\{A_1, A_2, \ldots, A_n\} \models A.$$  

If $A$ and $B$ are formulas and $A \models B$ we say that $A$ (tauto)logically implies $B$.  

Propositional Language - Semantics

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**Tautological equivalence**

**Definition:** For two formulas we write

\[ A \models B \]

to denote “\( A \models B \) and \( B \models A \).”

\( A \) and \( B \) are said to be **tautologically equivalent** (or simply equivalent) iff \( A \models B \) holds.

Tautologically equivalent formulas are assigned the same truth values by any value assignment.
Note that “(tauto)logically implies” is different from “implies”.

\[ A \models B \] if and only if \( A \rightarrow B \) is a tautology.

\( A \rightarrow B \) is a formula, which can be true or false.

Note that “(tauto)logically equivalent”, \( A \vDash B \), is different from “equivalent”.

\[ A \vDash B \] if and only if \( A \leftrightarrow B \) is a tautology.

\( A \leftrightarrow B \) is a formula, which can be true or false.
To prove a tautological consequence $\Sigma \vdash A$ we must show that any value assignment $v$ satisfying $\Sigma$ also satisfies $A$. One way to show this is by using truth tables.

Show that $\{A \rightarrow B, B \rightarrow C\} \vdash (A \rightarrow C)$

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The value assignments in rows 1, 2, 4, 8 are the only ones that satisfy $\Sigma = \{A \rightarrow B, B \rightarrow C\}$. For these value assignments also $A \rightarrow C$ is satisfied.

This shows that

$$\{A \rightarrow B, B \rightarrow C\} \models (A \rightarrow C),$$

This means that the argument

Premise 1: $A \rightarrow B$
Premise 2: $B \rightarrow C$

Conclusion: $A \rightarrow C$

is a valid argument.
When the argument is not valid

Prove that \((A \rightarrow B) \lor (A \rightarrow C) \not\models A \rightarrow (B \land C)\).

Find at least one row in the truth table for which the premises are true but the conclusion is false.
Size of truth tables

If the formula has $n$ variables and $m$ connectives

- How many rows does the truth table have? $2^n$
- How many columns does the truth table have? $\leq n + m$

We need another method for proving argument validity when the number of variables is too large.
We use “proof by contradiction” (here the word “contradiction” has a different meaning from the use of the word “contradiction” in logic, where it means a logical formula that is always false.)

Show that \( \{ A \rightarrow B, B \rightarrow C \} \models (A \rightarrow C) \).

**Proof:** Assume the contrary, that is \( \{ A \rightarrow B, B \rightarrow C \} \not\models (A \rightarrow C) \).

This means that there is a value assignment \( v \) such that

1. \( v(A \rightarrow B) = 1 \),
2. \( v(B \rightarrow C) = 1 \),
3. \( v(A \rightarrow C) = 0 \).

By (3) we have (4) \( v(A) = 1 \), and (5) \( v(C) = 0 \).

By (1) and (4) we have \( v(B) = 1 \).

From \( v(B) = 1 \) and (2) we have \( v(C) = 1 \) which contradicts (5).

As we reached a contradiction, our assumption must have been false, hence the argument is valid.
Example

A patient is administered three different tests.

Test $A$ will give a positive result if and only if either virus $X$ or virus $Y$ is present. Test $B$ will give a positive result if and only if virus $Y$ or $Z$ is present. If test $C$ is positive, virus $Y$ can be excluded. The patient reacts positively to all three tests.

Prove that he has virus $X$ and virus $Z$ but not virus $Y$. 

To prove \( \Sigma \not\models A \), we must construct a counter-example: A value assignment \( v \) satisfying \( \Sigma \) but not satisfying \( A \).

Show that \( \{ (A \rightarrow \neg B) \lor C, B \land \neg C, A \leftrightarrow C \} \not\models (\neg A \land (B \rightarrow C)) \).

**Proof:** Provide a counter-example.

Let \( v \) be the value assignment \( v(A) = 0 \), \( v(B) = 1 \) and \( v(C) = 0 \). Then we have
\[
\begin{align*}
v((A \rightarrow \neg B) \lor C) &= 1 \\
v(B \land \neg C) &= 1. \\
v(A \leftrightarrow C) &= 1 \\
v(\neg A \land (B \rightarrow C)) &= 0.
\end{align*}
\]

We found a value assignment that makes all premises true but the conclusion false, hence the argument is invalid.
Consider the following two statements:

He is either not informed, or he is not honest.
It is not true that he is informed and honest.

Intuitively, these two statements are logically equivalent. We prove this now. Define $p$ and $q$ to be the statements that “he is honest” and that “he is well informed” respectively.

The first statement translates into $\neg p \lor \neg q$, whereas the second into $\neg(p \land q)$. 
De Morgan’s Laws

De Morgan’s law: \( \neg(p \land q) \equiv (\neg p \lor \neg q) \)
which means that \( \neg(p \land q) \leftrightarrow (\neg p \lor \neg q) \) is a tautology.

<table>
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<tr>
<th>p</th>
<th>q</th>
<th>p \land q</th>
<th>\neg(p \land q)</th>
<th>\neg p \lor \neg q</th>
<th>\neg(p \land q) \leftrightarrow (\neg p \lor \neg q)</th>
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To negate a conjunction, take the disjunction of the negations of the conjuncts.

Dual De Morgan’s Law: \( \neg(p \lor q) \equiv (\neg p \land \neg q) \)
To negate a disjunction, take the conjunction of the negations of the disjuncts.
Consider the following pair of statements:

If the goods were not delivered, the customer cannot have paid.
If the customer has paid, the goods must have been delivered.

If \( q \) and \( p \) stand for “goods were delivered” and “customer paid” respectively, then these two statements translate into \( \neg q \to \neg p \) and \( p \to q \).
\( \neg q \to \neg p \) and \( p \to q \) are contrapositives of each other.
Contrapositives

The table below shows that contrapositives are equivalent, that is

\[ p \rightarrow q \equiv \neg q \rightarrow \neg p \]

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<th>\neg p</th>
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Sometimes in a proof it is easier to show that \( \neg q \rightarrow \neg p \) instead of showing that \( p \rightarrow q \).
Rene Descartes (1596-1650), French philosopher, famous for his statement “I think, therefore I am.”

Rene Descartes is sitting in a bar, having a drink. The bartender asks him if he would like another.

“I think not,” he says, and vanishes in a puff of smoke.

What is wrong with this joke?
Consider the following two statements:
(1) $p$ and $q$ have the same truth value
(2) If $p$, then $q$, and if $q$ then $p$.

The first statement becomes $p \leftrightarrow q$ and the second one $(p \rightarrow q) \land (q \rightarrow p)$.

The table below shows that $p \leftrightarrow q \models (p \rightarrow q) \land (q \rightarrow p)$

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<th>$p$</th>
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Proving a logical equivalence amounts to proving two logical implications.
Lemma If $A \models A'$ and $B \models B'$, then

1. $\neg A \models \neg A'$.
2. $A \land B \models A' \land B'$.
3. $A \lor B \models A' \lor B'$.
4. $A \rightarrow B \models A' \rightarrow B'$.
5. $A \leftrightarrow B \models A' \leftrightarrow B'$.
Replaceability of Equivalent Formulas and How to Negate a Formula

**Theorem.** Let $B \models C$. If $A'$ results from $A$ by replacing some (not necessarily all) occurrences of $B$ in $A$ by $C$, then $A \models A'$.

**Proof** By structural induction.

**Theorem.** Suppose $A$ is a formula composed by atoms and the connectives $\neg$, $\lor$, $\land$ only, by the formation rules concerned. Suppose that $\Delta(A)$ results by replacing in $A \land$ with $\lor$, $\lor$ with $\land$, and each propositional variable with its negation. Then $A \models \neg \Delta(A)$.

**Proof** By structural induction.

**Example.** Let $A = (p \land \neg q) \land (\neg r \land s)$. Find the negation of $A$, i.e. find $\neg A$. 