Semantics of Propositional Logic

1. Definition. The value of a formula \( \varphi \) under a truth valuation \( t \) is defined from the formation of \( \varphi \) as a well-formed formula, as follows.

\[
\begin{align*}
    p^t &= t(p) \\
    \neg \alpha^t &= \begin{cases} 
        1 & \text{if } \alpha^t = 0 \\
        0 & \text{if } \alpha^t = 1 
    \end{cases} \\
    \alpha \land \beta^t &= \begin{cases} 
        1 & \text{if } \alpha^t = \beta^t = 1 \\
        0 & \text{otherwise}
    \end{cases} \\
    \alpha \lor \beta^t &= \begin{cases} 
        1 & \text{if } \alpha^t = 1 \text{ or } \beta^t = 1 \\
        0 & \text{otherwise}
    \end{cases} \\
    \alpha \rightarrow \beta^t &= \begin{cases} 
        1 & \text{if } \alpha^t = 0 \text{ or } \beta^t = 1 \\
        0 & \text{otherwise}
    \end{cases}
\end{align*}
\]

(The property that the values of the parts determine the value of the whole is called compositionality.)

A formula has a finite number of variables, we can list all possible values in a finite truth table. For a formula with \( n \) variables, the full truth table has \( 2^n \) lines.

2. Example. The truth table of \( (p \lor q) \rightarrow (q \land r) \):

\[
\begin{array}{c|c|c|c|c}
p & q & r & (p \lor q) & (q \land r) & (p \lor q) \rightarrow (q \land r) \\
\hline
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Equivalence

3. Definition. Two formulas \( \alpha \) and \( \beta \) are equivalent, denoted \( \alpha \equiv \beta \), iff for every truth valuation \( v \), \( \alpha^v = \beta^v \).

Equivalent formulas are equivalent in any context. (A consequence of compositionality.)

4. Lemma. Suppose that \( \alpha \equiv \beta \). Then for any formula \( \gamma \), and any connective \( \ast \), the formulas \( \alpha \ast \gamma \) and \( \beta \ast \gamma \) are equivalent:

\[
\alpha \ast \gamma \equiv \beta \ast \gamma .
\]

Proof idea: a value \( (\alpha \ast \gamma)^v \) depends only on the values \( \alpha^v \) and \( \gamma^v \), and the identity of \( \ast \). Details left to you.
5. **Definition.** A formula $\alpha$ is *valid* if and only if for every truth valuation $t$, $\alpha^t = 1$. For propositional formulas, a valid formula is also called a *tautology*.

*Example:* $p \rightarrow p$.

A formula $\alpha$ is *satisfiable* if and only if there is some truth valuation $t$ such that $\alpha^t = 1$.

*Example:* $p \land q$.

A formula $\alpha$ is a *contradiction* if and only if for every truth valuation $t$, $\alpha^t = 0$.

*Example:* $p \land \neg p$.

**Note:**
- A formula is satisfiable iff it is not a contradiction.
- Any tautology is equivalent to $p \rightarrow p$.
- Any contradiction is equivalent to $p \land \neg p$.

The notion of satisfiability extends to sets of formulas.

6. **Definition.** Let $\Sigma$ denote a set of formulas and $t$ a valuation. Define

$$\Sigma^t = \begin{cases} 
1 & \text{if for each } \beta \in \Sigma, \beta^t = 1 \\
0 & \text{otherwise}
\end{cases}$$

When $\Sigma^t = 1$, we say that $t$ *satisfies* $\Sigma$.

A set $\Sigma$ is *satisfiable* iff there is some valuation $t$ such that $\Sigma^t = 1$.

7. **Definition.** Let $\Sigma$ be a set of formulas, and let $\alpha$ be a formula. We say that

- $\alpha$ is a *logical consequence* of $\Sigma$, or
- $\Sigma$ *(semantically) entails* $\alpha$, or
- in symbols, $\Sigma \vdash \alpha$,

if and only if for any truth valuation $t$,

$$\text{if } \Sigma^t = 1 \text{ then also } \alpha^t = 1.$$ 

We write $\Sigma \not\vdash \alpha$ for “not $\Sigma \vdash \alpha$”. That is, there exists a truth valuation $t$ such that $\Sigma^t = 1$ and $\alpha^t = 0$.

8. **Example.**

- $\{(p \rightarrow q), (q \rightarrow r)\} \vdash (p \rightarrow r)$.
- $\{(p \rightarrow \neg q) \lor r, (q \land \neg r), p \rightarrow r\} \not\vdash p \land (q \rightarrow r)$.

9. **Exercise.**

- $\emptyset \vdash \alpha$ means that $\alpha$ is a tautology. Why?
- $\{\alpha, \neg \alpha\} \vdash \beta$ is always true, whatever $\alpha$ and $\beta$ are. Why?
Equivalence can be expressed using the notion of entailment.

10. Lemma. $\alpha \equiv \beta$ if and only if both $\{\alpha\} \models \beta$ and $\{\beta\} \models \alpha$.


Algebraic transformations and normal forms

The propositional connectives satisfy the following “laws of Boolean algebra”.

<table>
<thead>
<tr>
<th>Commutativity:</th>
<th>DeMorgan’s Laws:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a. $\alpha \lor \beta \equiv \beta \lor \alpha$</td>
<td>7a. $\neg(\alpha \lor \beta) \equiv \neg\alpha \land \neg\beta$</td>
</tr>
<tr>
<td>1b. $\alpha \land \beta \equiv \beta \land \alpha$</td>
<td>7b. $\neg(\alpha \land \beta) \equiv \neg\alpha \lor \neg\beta$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Associativity:</th>
<th>Identity:</th>
</tr>
</thead>
<tbody>
<tr>
<td>2a. $(\alpha \lor \beta) \lor \gamma \equiv \alpha \lor (\beta \lor \gamma)$</td>
<td>8a. $\alpha \lor 0 \equiv \alpha$</td>
</tr>
<tr>
<td>2b. $(\alpha \land \beta) \land \gamma \equiv \alpha \land (\beta \land \gamma)$</td>
<td>8b. $\alpha \land 1 \equiv \alpha$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Distributivity of $\land$ over $\lor$:</th>
<th>Absorption</th>
</tr>
</thead>
<tbody>
<tr>
<td>3a. $(\alpha \land \beta) \lor \gamma \equiv (\alpha \lor \gamma) \land (\beta \lor \gamma)$</td>
<td>9a. $\alpha \lor 1 \equiv 1$</td>
</tr>
<tr>
<td>3b. $\alpha \lor (\beta \lor \gamma) \equiv (\alpha \lor \beta) \land (\alpha \lor \gamma)$</td>
<td>9b. $\alpha \land 0 \equiv 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Distributivity of $\lor$ over $\land$:</th>
<th>Contradiction:</th>
</tr>
</thead>
<tbody>
<tr>
<td>4a. $(\alpha \lor \beta) \land \gamma \equiv (\alpha \land \gamma) \lor (\beta \land \gamma)$</td>
<td>10. $\alpha \land \neg\alpha \equiv 0$</td>
</tr>
<tr>
<td>4b. $\alpha \land (\beta \lor \gamma) \equiv (\alpha \land \beta) \lor (\alpha \land \gamma)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Idempotence:</th>
<th>Excluded Middle:</th>
</tr>
</thead>
<tbody>
<tr>
<td>5a. $\alpha \lor \alpha \equiv \alpha$</td>
<td>11. $\alpha \lor \neg\alpha \equiv 1$</td>
</tr>
<tr>
<td>5b. $\alpha \land \alpha \equiv \alpha$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Self-inverse of $\neg$:</th>
<th>Implication:</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. $\neg\neg\alpha \equiv \alpha$</td>
<td>12a. $\alpha \rightarrow \beta \equiv \neg\alpha \lor \beta$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Contrapositive:</th>
</tr>
</thead>
<tbody>
<tr>
<td>12b. $\neg\alpha \rightarrow \neg\beta \equiv \beta \rightarrow \alpha$</td>
</tr>
</tbody>
</table>

13. $\alpha \rightarrow \beta \equiv \neg\beta \rightarrow \neg\alpha$

Proof. Check the truth tables.

12. Exercise. Prove or disprove each of the following.
1. \((p \land q) \lor (q \land r) \equiv q \land (p \lor r)\)
2. \((p \lor r) \land (q \lor s) \equiv (p \land q) \lor (p \land s) \lor (r \land q) \lor (r \land s)\)
3. \(\neg(p \lor \neg(r \lor s)) \equiv (p \land r) \lor (p \land s)\)
4. \(\neg(p \land q) \lor p \equiv 0\)
5. \(p \equiv p \land (q \Rightarrow p)\)
6. \(p \equiv p \land (\neg q \land \neg p) \lor p)\)
7. \(p \land (\neg q \land \neg p) \lor p) \equiv q\)

13. **Example (thanks to P. van Beek).**

Consider the following “logic puzzle” We are given a deck of cards, in which each card has a letter on one side and a natural number on the other side.

**Claim:** In these four cards, each card that has a vowel on one side has an even number on the other side.

\[
\begin{array}{c}
\text{A} \\
\text{B} \\
\text{\heartsuit} \\
\text{\heartsuit} \\
\text{\heartsuit} \\
\text{\heartsuit}
\end{array}
\]

You wish to determine whether this claim is true, with the least physical effort.

**Question:**

How many cards must you turn over, in order to determine whether or not the claim is true? Which ones?

An approach via propositional logic.

Choose variables; perhaps these.

- \(v_i\): card \(i\) has a vowel.
- \(e_i\): card \(i\) has an even number.

The picture shows that \(v_1\) is true, \(v_2\) is false, \(e_3\) is true, and \(e_4\) is false.

The claim asserts that \(v_i \Rightarrow e_i\) holds for each \(i\); that is, the formula

\[(v_1 \Rightarrow e_1) \land (v_2 \Rightarrow e_2) \land (v_3 \Rightarrow e_3) \land (v_4 \Rightarrow e_4)\]

is true.

Q: Does the picture justify the claim? That is, does

\[\{v_1, \neg v_2, e_3, \neg e_4\} \models (v_1 \Rightarrow e_1) \land (v_2 \Rightarrow e_2) \land (v_3 \Rightarrow e_3) \land (v_4 \Rightarrow e_4)?\]

A: No. For example, if \(e_1\) is false, then \((v_1 \Rightarrow e_1)\) is false.

However, we do have

\[\{v_1, \neg v_2, e_3, \neg e_4\} \models (v_2 \Rightarrow e_2) \land (v_3 \Rightarrow e_3).\]
**Answer to the puzzle:** turn over cards 1 and 4. If card 1 has an even number, and card 4 has a consonant, then the claim is true; otherwise it is false.

This answer is correct if and only if

\[ \{v_1, \neg v_2, e_3, \neg e_4\} \models (v_1 \to e_1) \land (v_2 \to e_2) \land (v_3 \to e_3) \land (v_4 \to e_4) \leftrightarrow (e_1 \land \neg v_4). \]

Alternatively, it is correct if and only if

\[ (1 \to e_1) \land (0 \to e_2) \land (v_3 \to 1) \land (v_4 \to 0) \equiv e_1 \land \neg v_4. \]

One can easily confirm these, either via truth tables or the algebraic identities.

**Duality**

Looking at the laws that involve \( \land \) and \( \lor \), we see a kind of symmetry. This illustrates a fundamental aspect of classical logic known as “duality”. Consider the following truth table, for a connective ‘?’:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p ? q</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

You immediately recognize it as the table for “and” (conjunction). Imagine, however, that one exchanges the meanings of the symbols 0 and 1; that is, use ‘0’ to mean “true” and ‘1’ to mean “false”. Now look back at the table. It now describes “or” (disjunction)!

14. **EXERCISE.** Does \( \to \) have a “dual” operation? Look at Rule 12a; the dual would satisfy

\[ \neg \alpha \star \neg \beta \equiv \alpha \land \neg \beta, \]

for some operation \( \star \).

a. What operation should \( \star \) be, in order to have a correct rule? (If you don’t know a name for it, just give its truth table.)

b. With your choice of \( \star \), does the “dual” of Rule 12b hold? In other words, is \( \alpha \star \beta \) equivalent to \( \neg \beta \star \neg \alpha \)?

**Alternative Sets of Connectives**

The formulas \( \alpha \to \beta \) and \( \neg \alpha \lor \beta \) are equivalent.

Thus \( \to \) is said to be *definable* in terms of \( \neg \) and \( \lor \).

We never need to use \( \to \); we can always write an equivalent formula without it. That is, we have defined more connectives than we need.

What about the opposite? Are there “formulas” that we cannot write, because we don’t have the necessary connectives?
Fix $n \in \mathbb{N}$, and propositional variables $p_1, p_2, \ldots, p_n$. A valuation gives a value to each of them; thus there are $2^n$ possible valuations—i.e., $2^n$ lines in a truth table. A column of the table has either 1 or 0 in each row; thus there are $2^2^n$ possible columns. We shall see that for each of them, there is a formula that realizes that pattern.

First, consider a column that has exactly one ‘1’ entry. (If there are none, the formula can be any contradiction; e.g., $p_1 \land \neg p_1$.) The row of this entry corresponds to an assignment $v$ given by $v(p_i) = a_i$, where each $a_i$ is either 1 or 0. For each $i$, let $f_i$ be $p_i$ if $a_i = 1$ and be $\neg p_i$ if $a_i = 0$. Let $g_v$ be the formula $f_1 \land f_2 \land \ldots \land f_n$. Then $g_v$ is true exactly at the row of $v$; that is, it is the desired formula.

Now consider a column with more than one 1 entry, say at rows $r_1, r_2, \ldots, r_k$. For each such row $r_i$, we found a formula $g_i$ above that is 1 only at that line. Then the formula $g_1 \lor \ldots \lor g_k$ is true at each of the lines, and no other. Thus it is the desired formula.

15. **Lemma.** Every truth table is realizable by a formula that uses only the connectives $\neg$, $\land$ and $\lor$.

16. **Definition.** A set of connectives is said to be *adequate* iff any $n$-ary ($n \geq 1$) connective can be defined in terms of the ones in the set.

Using this definition, we can rephrase the lemma: $\{\land, \lor, \neg\}$ is an adequate set of connectives.

17. **Lemma.** Each of the sets $\{\land, \neg\}$, $\{\lor, \neg\}$, and $\{\to, \neg\}$ is adequate.

18. **Exercise.** Prove the lemma. (Hint: for the first two, use De Morgan’s laws.)

Not every set, of course, is adequate.

19. **Lemma.** The set $\{\land, \lor\}$ is not an adequate set of connectives.

**Proof sketch.** Any formula that uses only $\land$ and $\lor$ has value 0 under the valuation that makes each variable false. (For full detail, do an induction on the number of connectives in the formula.)

20. **Exercise.**

1. Is there a connective $*$ such that the singleton set $\{*\}$ is adequate?

2. Are there connectives $c_1, c_2$ and $c_3$ such that $\{c_1, c_2, c_3\}$ is adequate, but none of $\{c_1, c_2\}$, $\{c_1, c_3\}$, or $\{c_2, c_3\}$ is adequate?

   (Such a set is called a *minimal* adequate set.)

3. Find all minimal adequate sets containing only binary, unary and nullary connectives.
Normal Forms

Looking back at the proof of Lemma 15 (the adequacy of \{\wedge, \vee, \neg\}), one sees that it actually proves more than the statement of the lemma. Not only do the formulas use only the given connectives, they do so in a restricted way.

- Every \neg applies to a variable, not to a larger formula.
- Every \wedge connects formulas that contain only \neg and \wedge as connectives.

21. Definition (Disjunctive and Conjunctive Normal Forms).

- A literal is either a variable or the negation of a variable.
- A formula \varphi is in Disjunctive Normal Form (DNF) iff it has the form \(D_1 \lor D_2 \ldots \lor D_k\), for some \(k \geq 1\), where each \(D_i\) is a conjunction of literals. Each \(D_i\) is a clause of \varphi.
- A formula \varphi is in Conjunctive Normal Form (CNF) iff it has the form \(C_1 \land C_2 \ldots \land C_k\), for some \(k \geq 1\), where each \(C_i\) is a disjunction of literals. Each \(C_i\) is a clause of \varphi.

(Note that the form of a “clause” depends on which normal form one is using. Confusing. Sorry about that.)

Some CNF formulas:

- \((p \lor \neg q) \land r \land (\neg r \lor p \lor q)\).
- \(\neg r \lor p \lor q\). (Only one clause.)
- \(\neg r \land p \land q\). (Three singleton clauses.)

Some formulas not in CNF:

- \(\neg(\neg p \land q)\). (\neg applied to \land.)
- \(p \lor (r \land q)\). (\lor applied to \land.)
- \(\neg \neg q\). (\neg applied to \neg.)
- \(p \rightarrow q\). (Uses \rightarrow.)

22. Lemma. Every formula is equivalent to a formula in Disjunctive Normal Form.

Proof. The same proof as for Lemma 15.

23. Lemma. Every formula is equivalent to a formula in Conjunctive Normal Form.

Proof. Two options: (1) Mimic the proof for DNF, except focus on the 0 entries rather than the 1 ones. (2) For a formula \varphi, first find a DNF formula \varphi′ equivalent to \neg \varphi. Then apply DeMorgan’s Laws to the formula \neg \varphi′, to move the negations in to apply to variables. The result is in CNF; also it is equivalent to \neg \neg \varphi and thus to \varphi.
 Conversion to normal form

Although the Boolean equivalences are symmetrical (one can substitute either side of the equivalence with the other side), the left-to-right directions permit conversion of any formula over the basis \{\neg, \land, \lor, \rightarrow\} to either conjunctive or disjunctive normal form. For each of the following steps, apply it as often as possible, and then continue to the next.

1. Replace every '→' using Rule 12a.
2. Where '→' applies to a compound formula, use Rule 6, 7a, or 7b, as applicable.
3. Now apply the appropriate distributive law.
   For conjunctive normal form: whenever '∨' applies to formulas containing '∧', use distributivity (rules 3a and 3b), as often as necessary.
   For disjunctive normal form, do the "dual" transformation: whenever '∧' applies to formulas containing '∨', apply rule 4a and 4b, as often as necessary.

In each case, applying a rule means to replace the left-hand side of the equivalence by the right-hand side.

24. Exercise. In following the procedure above, you may produce instances of \( p \lor p \), \( p \land p \), \( p \lor \neg p \), or \( p \land \neg p \), for some propositional variable \( p \). In each case, how can you simplify the formula?

 Working with CNF (or DNF) Formulas

In a clause, there is no need to have any variable twice. [Why?]
Also, the order of the literals in the clause does not matter.

Thus we can think of a clause as simply a set of literals.

Similarly, in a CNF formula, no clause [set] need appear more than once, and the order of clauses does not matter.

Thus we can think of a CNF formula as simply a set of clauses.

For example, the formula \( (p \lor q) \land (q \lor \neg r) \land s \) can be described by the set of clauses \{p, q\}, \{q, \neg r\} and \{s\}.

Special cases:

- In CNF, the empty clause is the disjunction of nothing; it is false. (It contains no satisfied literals.)
- In CNF, the empty formula (the set of zero clauses) is the conjunction of nothing; it is true. (Each of its clauses is satisfied.)