Due Thursday, Sept. 26, by 1:00 pm.

Note: see the course information for assignment guidelines, including the distinction between “main” and “supplementary” questions.

Main questions (total marks: 44)

Question 1 (8 marks).
In each part, give a resolution proof as indicated. Note that you may need preliminary work, before the resolution itself.

(a) \( \{ p \lor r, p \lor \neg r, \neg p \lor q \lor s, \neg q \lor s \} \vdash_{\text{Res}} s \)

(b) \( \{ \alpha, \beta \} \vdash_{\text{Res}} \bot \), where
\[
\alpha = (p \lor q \lor r) \land (\neg p \lor q \lor r) \land (p \lor \neg q \lor r) \land (\neg p \lor \neg q \lor r) \quad \text{and}
\beta = (p \lor q \lor \neg r) \land (\neg p \lor q \lor \neg r) \land (p \lor \neg q \lor \neg r) \land (\neg p \lor \neg q \lor \neg r).
\]

Question 2 (12 marks).
For this question, consider formulas over the basis \( \{ \land, \lor, \neg \} \). The size of a formula \( \varphi \), denoted \( |\varphi| \), is defined to be the number of occurrences of variables in \( \varphi \).

A truth assignment has even parity if an even number of variables receive value \( T \); it has odd parity if an odd number of variables receive value \( T \).

Let \( s_{\text{even}}(n) \) be the minimum size of a formula over \( n \) variables that gets value \( T \) iff the assignment has even parity. Let \( s_{\text{odd}}(n) \) be the minimum size of a formula over \( n \) variables that gets value \( T \) iff the assignment has odd parity.

(a) Show that \( s_{\text{even}}(1) = s_{\text{odd}}(1) = 1 \). Also show that for each \( n \geq 1 \), \( s_{\text{odd}}(n) = s_{\text{even}}(n) \).

(b) Show that for each \( n \geq 1 \), \( s_{\text{even}}(2n) \leq 4s_{\text{even}}(n) \).
   (Hint: suppose that you had formulas on \( n \) variables; use them to construct formulas for \( 2n \) variables.)

(c) Use the results from the previous parts to give an explicit upper bound on the size of parity formulas over \( \{ \land, \lor, \neg \} \).

Question 3 (12 marks).
Define that a CNF clause is redundant if it contains both a variable and its negation. Although such a clause is semantically a tautology, it can arise in a Resolution proof, which takes no account of semantics.

Prove the following, for every set \( \Sigma \) of CNF formulas (as premises): if there is a Resolution proof \( \Sigma \vdash_{\text{Res}} \bot \), then there is a Resolution proof of \( \Sigma \vdash_{\text{Res}} \bot \) that never uses a redundant clause in a resolution step.
Question 4 (12 marks).
Consider three sets of propositional variables, \( P = \{ p_i \mid i \geq 0 \} \), \( Q = \{ q_i \mid i \geq 0 \} \), and \( R = \{ r_i \mid i \geq 0 \} \). Suppose that formula \( \varphi_1 \) has variables from \( P \) and \( Q \) only, and formula \( \varphi_2 \) has variables from \( Q \) and \( R \) only.

Show that, if \( \varphi_1 \vdash \varphi_2 \), then there is a formula \( \psi \) with variables from \( Q \) only, such that \( \varphi_1 \vdash \psi \) and \( \psi \vdash \varphi_2 \).

Supplementary questions

Question 5 (12 marks).
(This question continues is related to Question 2; it differs in that it requires all formulas to be in CNF.)

(a) Let \( \varphi \) be a parity formula, in CNF; i.e., \( \varphi = C_1 \land \ldots \land C_\ell \), where each \( C_i \) is an \( \lor \) of literals.

Assume that no clause contains both a variable and its negation.

Show that each clause \( C_i \) must contain all \( k \) variables.

(b) Using the previous part (whether or not you solved it), show that a parity formula in CNF must have at least \( 2^{k-1} \) clauses.

Question 6 (12 marks).
Let there be \( k \) propositional variables \( p_1, p_2, \ldots, p_k \). Suppose that there are \( \ell \) distinct CNF clauses \( C_1, \ldots, C_\ell \) over these variables, that are mutually consistent (i.e., one cannot derive \( \bot \) by resolution).

In any one clause, each variable occurs at most once.

How large can \( \ell \) be, as a function of \( k \)?