Exercises are for your own use, as you see fit. Study on your own or with friends, ask questions in office hours and/or the Tutorial Centre, etc. Do as much or as little as you feel will be helpful.

Exercise 1. Show that the following predicate is undecidable:

On input $f$, for a recursive function $f$, determine whether $f(0)$ is defined.

Hint: show that if this predicate were decidable, then the Halting Problem would also be decidable.

Given $g$ and $x$, find $f$ s.t., $f(0)$ is defined if and only if $g(x)$ is defined.

Exercise 2. We separate the class of all recursive functions into “levels” $F_m$, for $m \geq 1$. Level $F_m$ contains the following functions.

- All of the base functions that have at most $m$ arguments.
- All functions definable from those base functions using at most $m$ applications of composition, primitive recursion and minimization.

(a) Confirm the following properties.

- Each $F_m$ is a finite set of recursive functions, with $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$.
- Every recursive function is in $F_m$, for some $m$ (and all larger $m$). Thus $\bigcup_{m \geq 1} F_m$ is the set of all recursive functions.

(b) For $m \geq 1$, let $B(m)$ be the maximum value attained as $f(x_1, \ldots, x_k)$, for some $f \in F_m$ and $x_i \leq m$ for each $i$. (In some cases, $f(\vec{x})$ is undefined and has no value. Take the maximum only over defined values.)

Explain why the value $B(m)$ exists, for every $m \geq 1$. That is, show that $B$ is a single, well-defined function. (Use the results of part (a).)

(c) Explain why $B$ is not a recursive function. That is, assume that $B$ is recursive, and derive a contradiction. (Use the results of part (a)—in a different way than in part (b)).

Exercise 3. Classical logic satisfies the “Contrapositive Law”: for any $\xi$ and $\eta$, $\xi \rightarrow \eta$ is equivalent to $\neg \eta \rightarrow \neg \xi$. In constructive logic, however, this fails in general.

In each part, state whether the specified deduction can be carried out constructively, for arbitrary $\xi$ and $\eta$. If so, give a constructive proof; if not, justify why no such proof can exist.

(a) Given $\xi \rightarrow \eta$, prove $\neg \eta \rightarrow \neg \xi$.
(b) Given $\neg \xi \rightarrow \eta$, prove $\neg \eta \rightarrow \xi$.
(c) Given $\xi \rightarrow \neg \eta$, prove $\eta \rightarrow \neg \xi$.
(d) Given $\neg \xi \rightarrow \neg \eta$, prove $\eta \rightarrow \xi$. 
Exercise 4.

(a) Show that the premise $\neg\neg p \rightarrow p$ does NOT suffice to prove $p \lor \neg p$ constructively.

(b) Both textbooks state, and justify, that the double-negation law implies excluded middle. How can this be, given the result of the previous part?