CS 335
Computational Methods in Business and Finance
Winter 2020
Lecture 3

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• In general, if we can buy 1 option for \( V^* < V_{\text{no-arbitrage}} \), we can lock in a profit of \( V_{\text{no-arbitrage}} - V^* \).

• Exercise: analyze the case where option value > 0.633.

• Note: no-arbitrage value (0.633) ≠ expected value (0.097)

• The no-arbitrage price is independent of the actual prob of the up/down movements of \( S \).

• Hedging strategy (choice of \( \delta \)) independent of \( p_{\text{up}}/p_{\text{down}} \).

**Option pricing**

• Does not attempt to predict underlying asset price movements.

• Basic assumptions: risky asset follows a “stochastic process”; i.e. they are random and unpredictable.

• Limitation of two-state tree: stock prices have a prob distribution of outcomes:
Brownian motion

- Suppose $X(t)$ is a random variable. In time $t \rightarrow t + dt$, $X \rightarrow X + dX$ and

$$dX = \alpha \ dt + \sigma \ dZ$$

increment of a Wiener process

where $dZ = \phi \sqrt{dt}$

$\phi = \text{random number drawn from a standard normal distribution}$

$\phi \sim N(0,1)$

- Standard normal density:

$$p(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$\text{prob}(a \leq y \leq b) = \int_a^b p(y) \, dy$$
• More generally, if a random variable \( y \sim N(\mu, \beta^2) \), the density is given by

\[
p(y) = \frac{1}{\sqrt{2\pi\beta^2}} e^{-\frac{(y-\mu)^2}{2\beta^2}}
\]

with \( E[y] = \text{mean/expectation} \)

\[
= \int_{-\infty}^{\infty} yp(y) \, dy
\]

\( = \mu \)

\( \text{var}[y] = \text{variance} \)

\[
= E[(y - E[y])^2]
\]

\( = E[y^2] - (E[y])^2 \)

\( = \beta^2 \)

• If \( y \sim N(0, 1) \), then \( E[y] = 0, \text{var}[y] = 1 \).

• Back to Brownian motion, we have

\[
dZ = \phi \sqrt{dt}, \quad \phi \sim N(0,1)
\]

\( \therefore \quad E[\phi] = 0 \)

\[
1 = \text{var}[\phi] = E[\phi^2] - (E[\phi])^2 = E[\phi^2]
\]
• Consider \( dX = \alpha \, dt + \sigma \, dZ \)

\[
E[dX] = E[\alpha \, dt] + E[\sigma \, dZ] \quad (E[dZ] = E[\phi \sqrt{dt}] = 0)
\]

\[
= \alpha \, dt
\]

\[
\text{var}[dX] = E[(dX - E[dX])^2]
\]

\[
= E[(dX - \alpha \, dt)^2]
\]

\[
= E[(\sigma \, dZ)^2]
\]

\[
= \sigma^2 \, E[\phi^2 \, dt]
\]

\[
= \sigma^2 \, dt
\]

Notes:

• \( dX = \alpha \, dt + \sigma \, dZ \) is a stochastic differential equation (SDE); the presence of a random (stochastic) term \( dZ \).

• No single solution to an SDE, only a prob density of outcome.
Lattice Model

Let $X(t_i) =$ position of a particle following Brownian motion with drift at time $t_i$.

At $t_0 = 0$, $X(t_0) = X_0$. Consider $t_1 = t_0 + \Delta t$
• We assume $X$ follows a Markov process:

• prob of an up/down move is indep of past history

• prob distribution of future value depends only on where we are now

Let $X(t_i) = X_i$ and $\Delta X_i = X_{i+1} - X_i$

Note: $\Delta X_i$ are independent and identically distributed (i.i.d.). $\Delta X_i$ and $\Delta X_j$ are pairwise independent ($i \neq j$)

i.e. $E[\Delta X_i \Delta X_j] = E[\Delta X_i] E[\Delta X_j]$

Now

\[ \Delta X_i \xrightarrow{p} + \Delta h \quad \xrightarrow{q} - \Delta h \]

$E[\Delta X_i] = p\Delta h + q(-\Delta h) = (p-q)\Delta h$

$E[\Delta X_i^2] = p(\Delta h)^2 + q(-\Delta h)^2 = (p+q)(\Delta h)^2 = (\Delta h)^2$

Note: results are independent of $i$. 
Then
\[ \text{var}[\Delta X_i] = E[(\Delta X_i)^2] - (E[\Delta X_i])^2 \]
\[ = (\Delta h)^2 - (p-q)^2 \Delta h^2 \]
\[ = (\Delta h)^2 [1 - (p-q)^2] \]
\[ = (\Delta h)^2 [1 - (p-(1-p))^2] \]
\[ = (\Delta h)^2 4p(1-p) \]
\[ = (\Delta h)^2 4pq \]

Suppose we consider a finite time $t$.

Total number of moves $= \frac{t}{\Delta t} = n$.

After $n$ steps,

\[ p[j \text{ up moves, } n-j \text{ down moves}] = \frac{n!p^jq^{n-j}}{j!(n-j)!} = \binom{n}{j} p^j q^{n-j} \]

Now
\[ X_n - X_0 = \sum_{i=0}^{n-1} \Delta X_i \]
\[ E[X_n - X_0] = E[\sum_{i=0}^{n-1} \Delta X_i] \]
\[ = \sum_{i=0}^{n-1} E[\Delta X_i] = \sum_{i=0}^{n-1} (p - q)\Delta h \]
\[ = n(p-q)\Delta h \]
\[ = \frac{t}{\Delta t} (p-q) \Delta h \]