CS 335
Computational Methods in Business and Finance
Winter 2020
Lecture 4

© Justin W.L. Wan (2020)
var[X_n - X_0] = var[\sum_{i=0}^{n-1} \Delta X_i]

= E[(\sum_i \Delta X_i - E[\sum_i \Delta X_i])^2]

= E[(\sum_i \Delta X_i - \sum_i E[\Delta X_i])^2]

= E[(\sum_i (\Delta X_i - E[\Delta X_i])^2) = E[(\sum_i a_i)^2]

= E[(\sum_i a_i)(\sum_j a_j)]

= E[\sum_{i,j} (a_i a_j)]

= E[\sum_{i=j} a_i^2 + \sum_{i \neq j} a_i a_j]

= \sum_i E[a_i^2] + \sum_{i \neq j} E[a_i a_j]

Note \ E[a_i a_j] = E[(\Delta X_i - E[\Delta X_i])(\Delta X_j - E[\Delta X_j])]

= E[\Delta X_i - E[\Delta X_i]]*E[\Delta X_j - E[\Delta X_j]]

(since \Delta X_i, \Delta X_j are indep)

= (E[\Delta X_i] - E[\Delta X_i])*(E[\Delta X_j] - E[\Delta X_j])

= 0

∴ var[X_n - X_0] = \sum_i E[a_i^2] = \sum_i E[(\Delta X_i - E[\Delta X_i])^2]

= \sum_i var[\Delta X_i]

= \sum_i 4pq(\Delta h)^2

= n4pq(\Delta h)^2 = \frac{t}{\Delta t} 4pq(\Delta h)^2
\textbf{Limit } \Delta t \to 0

- Want to determine \( \Delta h \) and \( p \).
- Let \( \Delta t \to 0 \), \( n \to \infty \), \( n\Delta t = T = \text{fixed} \).
- We will assume that \( \lim_{\Delta t \to 0} pq \neq 0 \) (otherwise random walk is not random!)
- Recall \( \text{var}[X_n - X_0] = \frac{t}{\Delta t} 4pq(\Delta h)^2 \)
- If \( \text{var}[X_n - X_0] = 0 \) or \( \infty \), then it would not be very interesting.
- Hence \( (\Delta h)^2 \propto \Delta t \Rightarrow \Delta h \propto \sqrt{\Delta t} \)

i.e. \( \Delta h = C_1 \sqrt{\Delta t} \), \( C_1 = \text{indep of } \Delta t \)

\( \Rightarrow \quad \text{E}[X_n - X_0] = \frac{t}{\Delta t} (p - q) C_1 \sqrt{\Delta t} = \frac{C_1 t(p-q)}{\sqrt{\Delta t}} \)

\( \text{var}[X_n - X_0] = \frac{t}{\Delta t} 4pq C_1^2 \Delta t = 4pq t C_1^2 \)

- Similarly, \( \text{E}[X_n - X_0] \) should be bounded and \( > 0 \) as well.

\( \Rightarrow \quad p-q \propto \sqrt{\Delta t} \Rightarrow p-q = C_2 \sqrt{\Delta t} \), \( C_2 = \text{indep of } \Delta t \)

\( \Rightarrow \quad \text{E}[X_n - X_0] = C_1 C_2 t \)
• Note: \( p+q = 1 \) and \( p-q = C_2 \sqrt{\Delta t} \)

=> \[ p = \frac{1}{2} \left( 1 + C_2 \sqrt{\Delta t} \right) \]

=> \[ q = \frac{1}{2} \left( 1 - C_2 \sqrt{\Delta t} \right) \]

• Recall \( \text{var}[X_n - X_0] = 4pq t C_1^2 \)

\[ = C_1^2 t \left( 1 - C_2^2 \Delta t \right) \]

Conclusion: as \( \Delta t \to 0 \),

\[ \text{E}[X_n - X_0] = C_1 C_2 t \]

\[ \text{var}[X_n - X_0] = C_1^2 t \quad C_1, C_2 \text{ indep of } \Delta t \]
Now suppose that

\[ X_n - X_0 \text{ is very small} \]

i.e. \[ X_n - X_0 \sim dX \]

and \([0, t]\) is also very small

i.e. \[ t \sim dt \]

Then \[ E[dX] = C_1 C_2 \ dt \]
\[ \text{var}[dX] = C_1^2 \ dt \]

Recall Brownian motion SDE:

\[ dX = \alpha \ dt + \sigma \ dZ \]
\[ E[dX] = \alpha \ dt \]
\[ \text{var}[dX] = \sigma^2 \ dt \]

If we try to match the first 2 moments (mean, variance) of the lattice random walk with the SDE,

\[ C_1 = \sigma, \quad C_2 = \alpha / \sigma \]

[Def: A solution of an SDE is an expression for the probability density of \(X(t)\).]
• It can be proved that the prob density of the random walk on the lattice, as \( \Delta t \to 0 \), converges to the solution to the SDE: \( dX = \alpha \, dt + \sigma \, dZ \) with \( C_1 = \sigma, \, C_2 = \frac{\alpha}{\sigma} \).

Note: \[
\lim_{\Delta t \to 0} \text{var}[X_n - X_0] = \text{var}[X(t)-X(0)] = \sigma^2 \, t
\]

\[
\lim_{\Delta t \to 0} E[X_n - X_0] = E[X(t)-X(0)] = \alpha \, t
\]

Since \( X_n - X_0 \) has a binomial distribution, and

\[
\lim_{\Delta t \to 0} (\text{binomial}) \to \text{normal}
\]

\[
\Rightarrow \quad X(t)-X(0) \sim N(\alpha \, t, \sigma^2 \, t)
\]

**Special case: \( \alpha = 0, \sigma = 1 \)**

\[dX = \alpha \, dt + \sigma \, dZ = dZ \] (\( Z(t) \) is a standard Brownian motion)

\[
\Rightarrow \quad Z(t) - Z(0) \sim N(0, t)
\]

We can write this as

\[
Z(t) - Z(0) = \phi \sqrt{t} \quad \text{where} \quad \phi \sim N(0,1)
\]

If \( t \sim dt, \, Z(0) = 0, \) then

\[
dZ = \phi \sqrt{dt} \quad \text{where} \quad \phi \sim N(0,1)
\]
Summary

Discrete random walk on lattice:

\[ X_{\hat{n}} \xrightarrow{p} X_{\hat{n}} + \Delta h \]

\[ X_{\hat{n}} \xrightarrow{q} X_{\hat{n}} - \Delta h \]

with properties:

\[ \Delta h = \sigma \sqrt{\Delta t} \]

\[ p = \frac{1}{2} \left( 1 + \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \]

\[ q = \frac{1}{2} \left( 1 - \frac{\alpha}{\sigma} \sqrt{\Delta t} \right) \]

converges as \( \Delta t \to 0 \) to the solution of SDE:

\[ dX = \alpha \, dt + \sigma \, dZ \]

\[ dZ = \varnothing \sqrt{dt} \]

\[ \varnothing \sim N(0,1) \]

This explains the scaling of \( dZ = \varnothing \sqrt{dt} \).
Properties of Brownian motion

As $\Delta t \to 0$, the distance travelled by a particle in a finite time becomes infinite.

Why? $|X_{i+1} - X_i| = |\Delta X| = \Delta h$

Total distance travelled in time $t$:

$$n |\Delta X| = \left(\frac{t}{\Delta t}\right) \Delta h = \left(\frac{t}{\Delta t}\right) \sigma \sqrt{\Delta t}$$

$$= \frac{t \sigma}{\sqrt{\Delta t}} \to \infty \quad \text{as } \Delta t \to 0$$

• Similarly,

$$\frac{\Delta X}{\Delta t} = \pm \frac{\sigma \sqrt{\Delta t}}{\Delta t} \to \pm \infty \quad \text{as } \Delta t \to 0$$

• Paths not differentiable

• Needs to use stochastic calculus