Assignment 5 (due Friday, April 3rd, 6:00pm)

Instructions:

• Hand in your assignment using Crowdmark. Detailed instructions are on the course website.

• Give complete legible solutions to all questions.

• Your answers will be marked for clarity as well as correctness.

• For any algorithm you present, you should justify its correctness (if it is not obvious) and analyze the complexity.

1. [15 marks] Farmer has fields that are modeled as squares on $n \times n$ two-dimensional grid. The grid of cells has $n$ rows going from south to north and numbered from 1 to $n$, and $n$ columns going from west to east numbered from 1 to $n$. So coordinates of south-west corner are $(1, 1)$, and coordinates of north-east corner are $(n, n)$. Each grid cell either contains corn or water. This data is represented as array $F$ of characters, with $F_{ij}$ equal to “c” or “w” (representing “corn” or “water” respectively).

A fire starts in a corn cell $(x, y), 1 \leq x, y \leq n$ at time 0, and simultaneously spreads one cell in each direction (North, South, East, West) in each unit of time (unless the cell was already burned or contains water). It also take one unit of time for the cell of corn to be completely burned. The fire cannot spread to water (and thus can be blocked by water).

Describe a graph representation of this problem and answer the following question by providing an algorithm (based on standard algorithms considered in class) and run-time analysis in terms of $n$. In particular, if you build a graph with $n^2$ vertices ($V = \{1, 2, \ldots, n^2\}$) give an explicit construction of set of edges $E$. Provide a pseudocode description of an algorithm CONVERTTOGRAPH that takes $F$, and $n$ as parameters and returns graph (a pair $(V, E)$) in your favourite representation.

(a) will all corn burn?

Provide a pseudocode description of an algorithm WILLITBURN that takes $F$ and coordinates $x, y$ as parameters and returns TRUE or FALSE.

(b) how long will it take for the fire to be over?

Provide a pseudocode description of an algorithm TIMETOburn that takes $F$ and a coordinates $x, y$ as parameters and returns integer value which represents the time the fire is over.

(c) if not all corn will burn, how many cells with corn will remain?

Provide a pseudocode description of an algorithm SAVERFIELDS that takes $F$ and a coordinates $x, y$ as parameters and returns the number of cells left unharmed.
2. [10 marks] Show that the following problems are in NP by describing a polynomial size certificate and polynomial time verification algorithm.

(a) **k-Path**: Given a directed graph $G = (V, E)$ and $s, t \in V$, and $k$, does there exist a directed path of length at least $k$ from $s$ to $t$ in $G$?
(b) **Double-Clique**: Given graph $G$ and $k$, does $G$ contain at least $2$ $k$-cliques?

3. [10 marks] Show that the following problems are in P by describing a polynomial time decision algorithm.

(a) **4-Path**: Given a directed graph $G = (V, E)$ and $s, t \in V$, does there exist a directed path of length 4 from $s$ to $t$ in $G$?
(b) **2-Clique**: Given graph $G$, does there exist a clique of size 2 in $G$?

4. [30 marks] Given a directed graph with positive edge weights and a specified vertex $v$ in the graph, the “all-pairs $v$-constrained shortest path problem” is the problem of computing, for each pair of vertices $i$ and $j$, the length of the shortest path from $i$ to $j$ that goes through the vertex $v$. If $i = v$ or $j = v$, the constraint is already satisfied. If no such path exists, the answer is $\infty$. Note that a $v$-constrained shortest path might contain repeated vertices.

(a) Describe an algorithm that takes a graph $G = (V, E)$ and vertex $v$ as input parameters and computes values $L(i, j)$ that represent the length of the $v$-constrained shortest path from $i$ to $j$ for all $1 \leq i, j \leq |V|$. Provide a brief justification of its correctness. Your algorithm should have a running time in $O(|V|^2)$.
(b) Marmoset: Implement your algorithm taking the following into consideration.

The input consists of several lines: first line has two natural numbers $n$ – the number of vertices in the given graph and $v$, $1 \leq v \leq n$ – specified vertex. It is followed by $n$ lines, where line $i$ contains a list of vertices adjacent to the vertex $i$ and weights of corresponding edges.

The output consists of $n$ lines, where each line contains $n$ numbers: Number in $i$th row and $j$th column represents the length of the $v$-constraint shortest path from vertex $i$ to vertex $j$. Use $-1$ to represent $\infty$.

**Input example:**

```
5 3
3 2 4 1
1 4 3 17 5 3
4 12
5 1
1 2
```

**Output example:**

```
17 -1 2 14 15
21 -1 6 18 19
15 -1 0 12 13
20 -1 5 17 18
19 -1 4 16 17
```

The example represents a graph with 5 vertices, vertex 1 is adjacent to vertices 3 and 4 (weights of edges are 2 and 1), vertex 2 is adjacent to vertices 1, 3 and 5 (weights of edges are 4, 17 and 3), vertex 3 is adjacent to vertex 4 (weight of edge is 12) and so on.

The specified vertex is vertex 3.

Please follow Piazza for implementation and submission details.