ASSIGNMENT 8

DUE: Wednesday November 20, 5 PM. NOTE that Crowdmark will list the due date as Friday November 22 but that ONLY applies if you are using a grace credit for this assignment.
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Please read http://www.student.cs.uwaterloo.ca/~cs341 for general instructions and policies. When you are asked to design an algorithm you should: (1) describe the idea of your algorithm clearly in English; (2) give pseudocode; (3) argue correctness; (4) and analyze the run time.

This assignment is about the hardness of the Shortest Simple Path problem (SSP) in its optimization and decision versions.

- **OptSSP problem:**
  **Input:** A directed graph $G$ with integer weights on the edges and two vertices $s$ and $t$.
  **Output:** The minimum weight of a simple path from $s$ to $t$ in $G$.

- **DecideSSP problem:**
  **Input:** A directed graph $G$ with integer weights on the edges, two vertices $s$ and $t$, and an integer $k$.
  **Output:** Is there a simple path from $s$ to $t$ of weight $\leq k$?

1. [6 marks] Show that the OptSSP and DecideSSP problems are equivalent with respect to polynomial-time algorithms. In other words, prove that:

   (i) **DecideSSP $\leq_P$ OptSSP.**
   
   *Hint.* This is very easy. Start by assuming you have a polynomial time algorithm for OptSSP...

   (ii) **OptSSP $\leq_P$ DecideSSP.**
   
   *Hint.* The obvious approach would be to run the algorithm for DecideSSP on all possible values of $k$ to find the smallest one. An overly generous bound on the range of $k$ is as follows: let $w = \max\{|w(e)| : e \text{ an edge of } G\}$. Then any simple path in $G$ has weight in the range from $-nw$ to $nw$. This range has $2nw + 1$ integer values, so we do not get a polynomial run time if we try all of them. However, note that $\log(2nw + 1)$ is $O(\log n + \log w)$ which is a polynomial in the input size.

2. [10 marks] The **HamiltonianPath** problem (for directed graphs) is defined as follows:
   **Input:** A directed graph $G$.
   **Output:** Is there a simple path in $G$ that visits every vertex exactly once?

   Prove that **HamiltonianPath $\leq_P$ DecideSSP.**

3. [5 marks] Prove that **DecideSSP $\in$ NP.**

   Specify the certificate (which must be of polynomial size) and the polynomial-time verification algorithm.

Your proofs for 2 and 3 show that DecideSSP is NP-complete, since HamiltonianPath is NP-complete.