CS 341: ALGORITHMS

Lecture 10: graph algorithms I

Readings: see website

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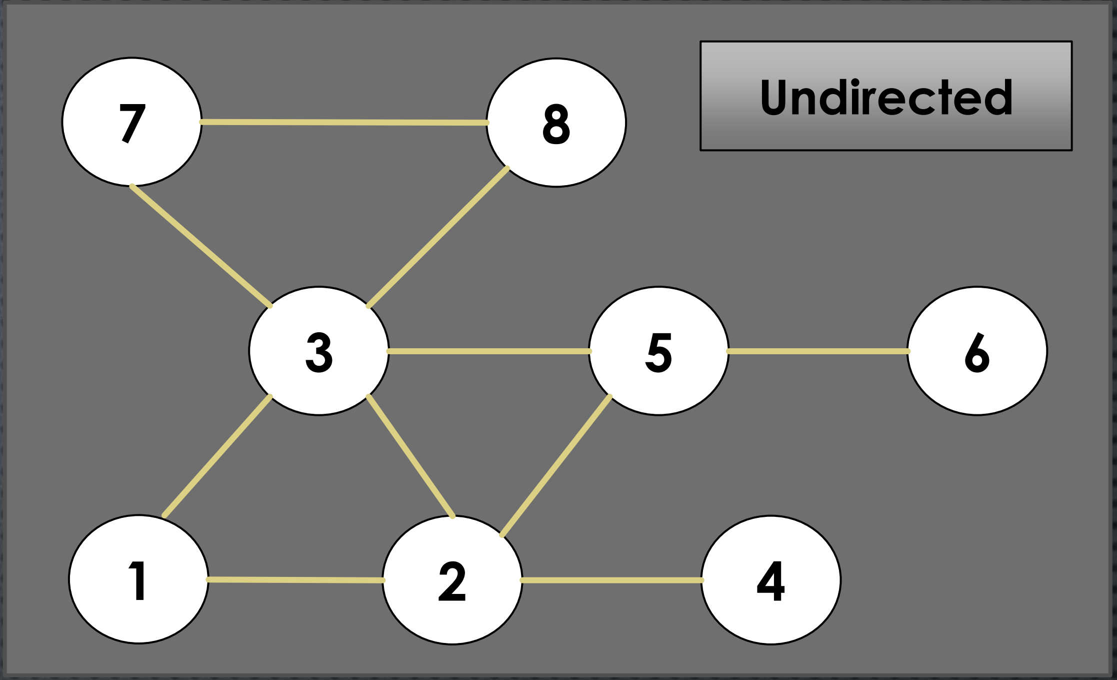
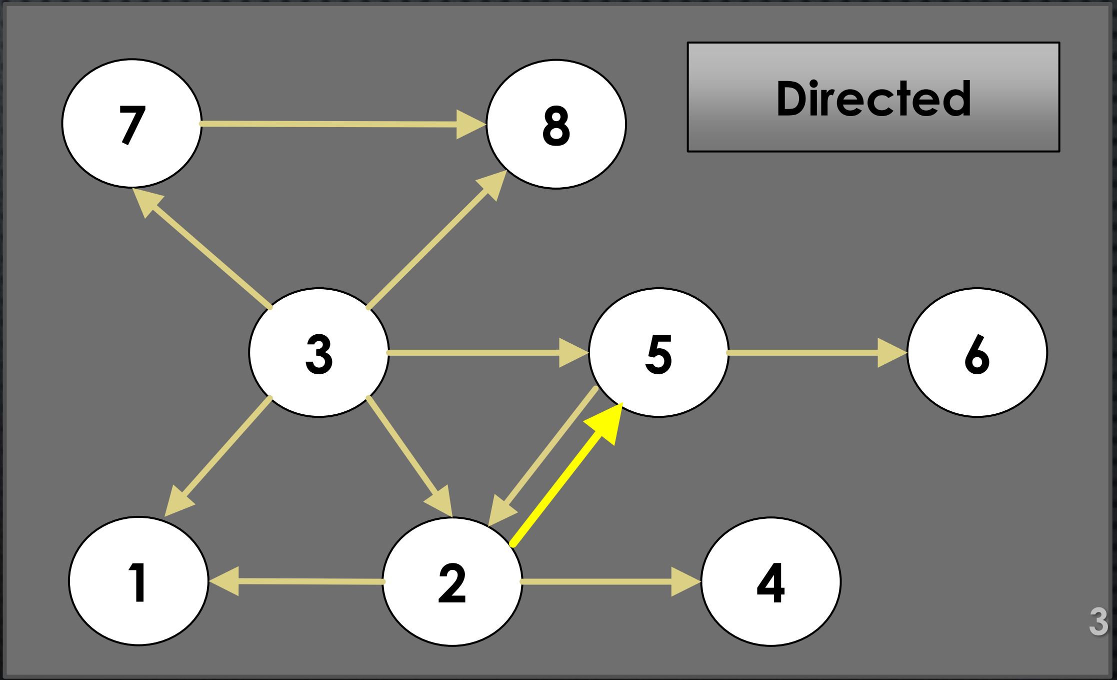
GRAPHS

GRAPHS

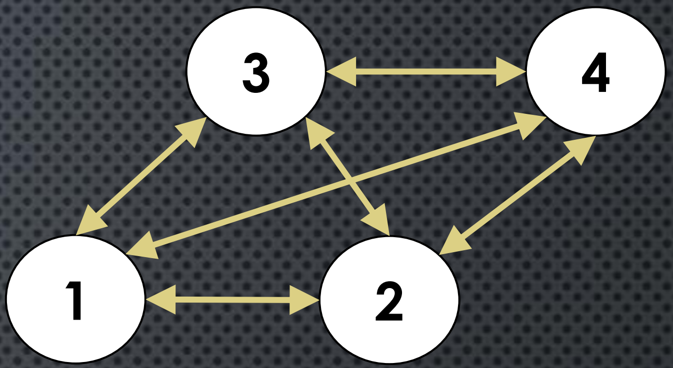
- A graph is a pair $G = (V, E)$
- V contains **vertices**
- E contains **edges**
  - An edge uv connects two **distinct** vertices u, v
  - Also denoted (u, v)
- Graphs can be **undirected**
- … or **directed**
  - meaning $(u, v) \neq (v, u)$





PROPERTIES OF GRAPHS

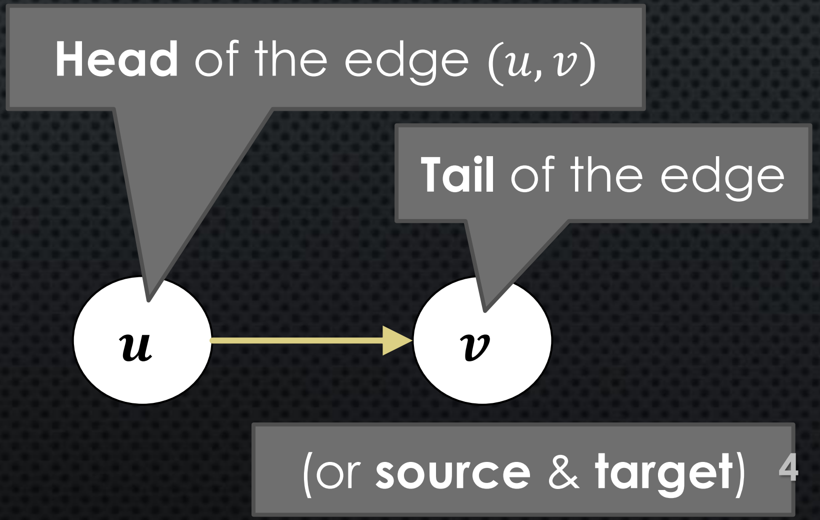
- Number of vertices $n = |V|$
- Number of edges $m = |E| \leq n(n-1)$
    - Note m is in $O(n^2)$ but **not necessarily** $\Omega(n^2)$
    - For undirected graphs, $m \leq \frac{n(n-1)}{2}$
        - (Asymptotically, no different)



12 edges

$n(n-1) = 4 \cdot 3$

- Other common terminology:
    - **vertices = nodes** edges = arcs



Head of the edge (u, v)

Tail of the edge

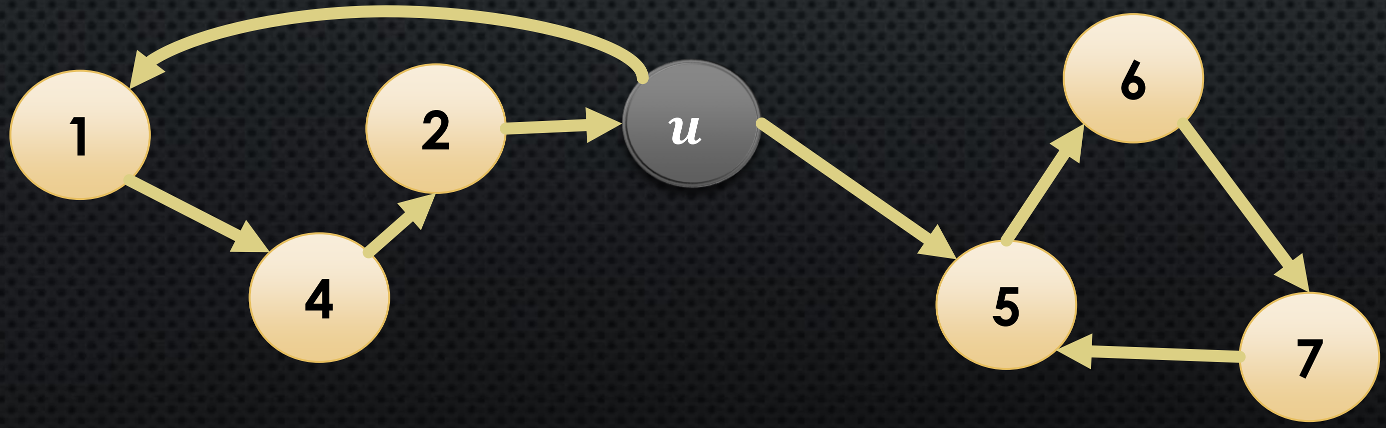
(or **source** & **target**)

A FEW MORE TERMS

- The **indegree** of a node $\boldsymbol{u}$, denoted $\mathrm{indeg}(u)$, is the number of edges **directed into** u
- The **outdegree**, denoted $\mathrm{outdeg}(u)$, is the number of edges **directed out from** u

or simply $\mathrm{deg}(u)$ in an undirected graph

- The **neighbours** of u are the nodes u points to
  - Also called the **nodes adjacent to $\boldsymbol{u}$**, denoted $\mathrm{adj}(u)$



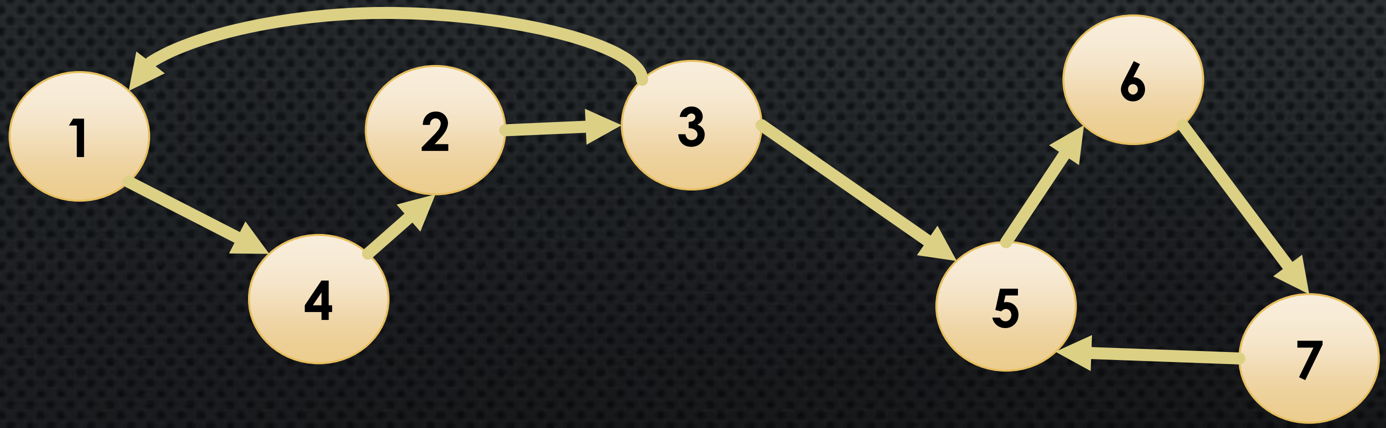
$$indeg(u) = 1$$
$$outdeg(u) = 2$$
$$adj(u) = \{1{,}5\}$$

DATA STRUCTURES FOR GRAPHS

- Two main representations
  - **Adjacency matrix**
  - **Adjacency list**
- Each has pros & cons

ADJACENCY MATRIX REPRESENTATION

- $n \times n$ matrix $A = (a_{uv})$
  - rows & columns indexed by V
- $\boldsymbol{a_{uv}} = \mathbf{1}$ if (u, v) **is an edge**
- $\boldsymbol{a_{uv}} = \mathbf{0}$ if (u, v) is a **non-edge**
- Diagonal = 0 (no self edges)

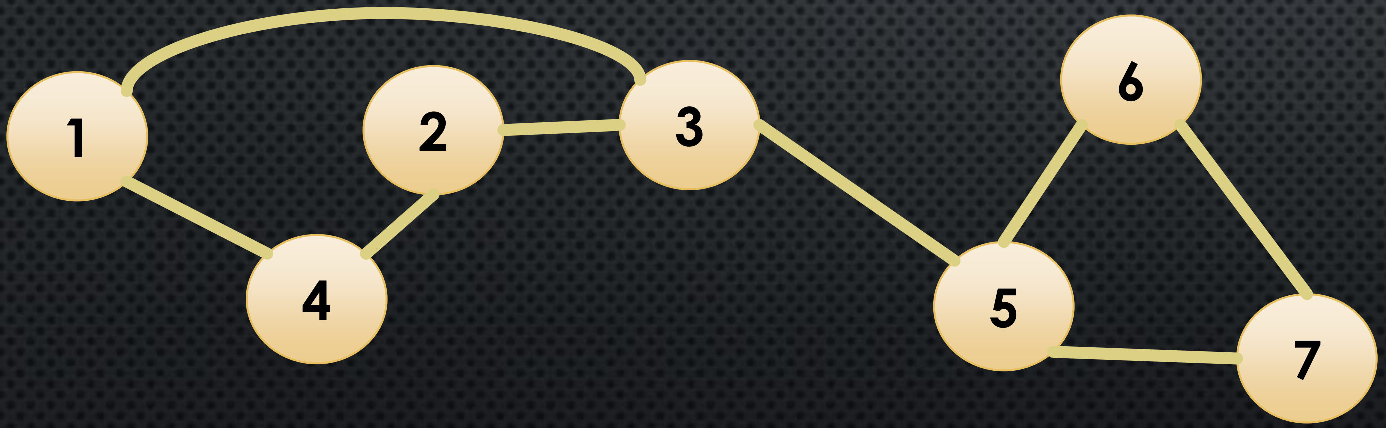


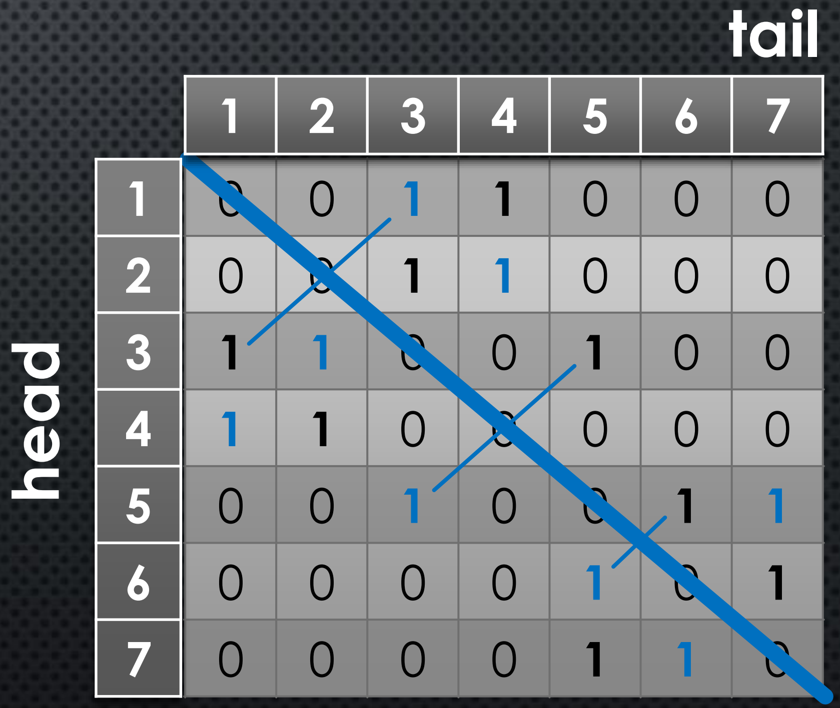
Matrix A

	1	2	3	4	5	6	7
1	0	0	0	**1**	0	0	0
2	0	0	**1**	0	0	0	0
3	**1**	0	0	0	**1**	0	0
4	0	**1**	0	0	0	0	0
5	0	0	0	0	0	**1**	0
6	0	0	0	0	0	0	**1**
7	0	0	0	0	**1**	0	0

ADJACENCY MATRIX REPRESENTATION

- **For undirected graphs**
- $\boldsymbol{a_{uv}} = \mathbf{1}$ if (u, v) **or** $\boldsymbol{(v, u)}$ is an edge
- Matrix is symmetric $A^T = A$





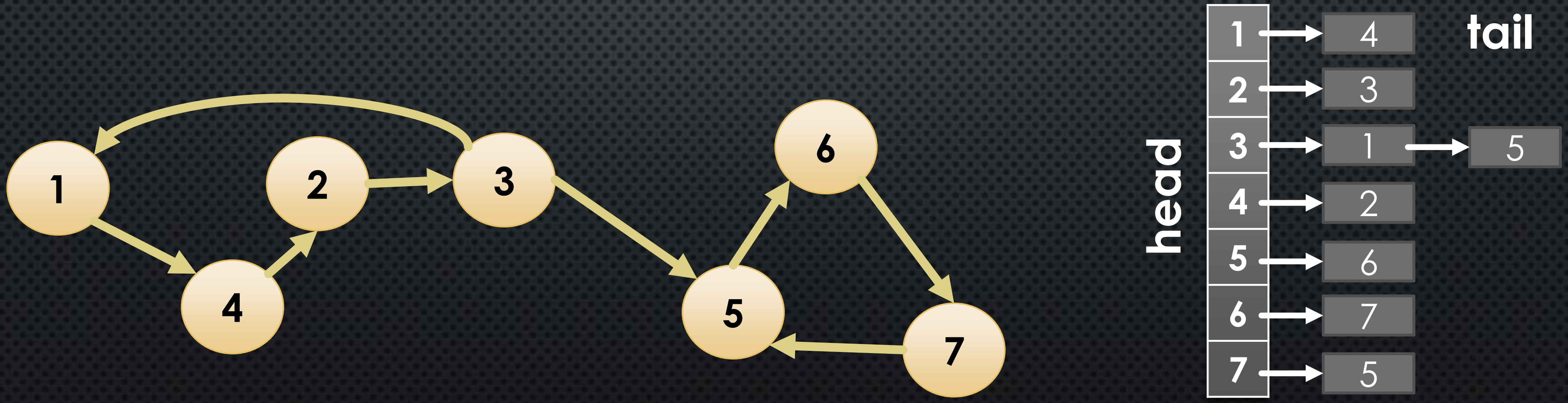
	1	2	3	4	5	6	7
1	0	0	1	1	0	0	0
2	0	0	1	1	0	0	0
3	1	1	0	0	1	0	0
4	1	1	0	0	0	0	0
5	0	0	1	0	0	1	1
6	0	0	0	0	1	0	1
7	0	0	0	0	1	1	0

IMPLEMENTING AN ADJACENCY MATRIX

- Suppose we are loading a graph from input
  - Assume nodes are labeled 0..n-1
  - 2D array **bool adj[n][n]**
- What if nodes are not labeled 0..n-1?
  - Rename them in a preprocessing step
- What if you don't have 2D arrays?
  - Transform 2D array index into 1D index
  - adj[u][v] → adj[u*n + v]
    (can simplify with macros in C)

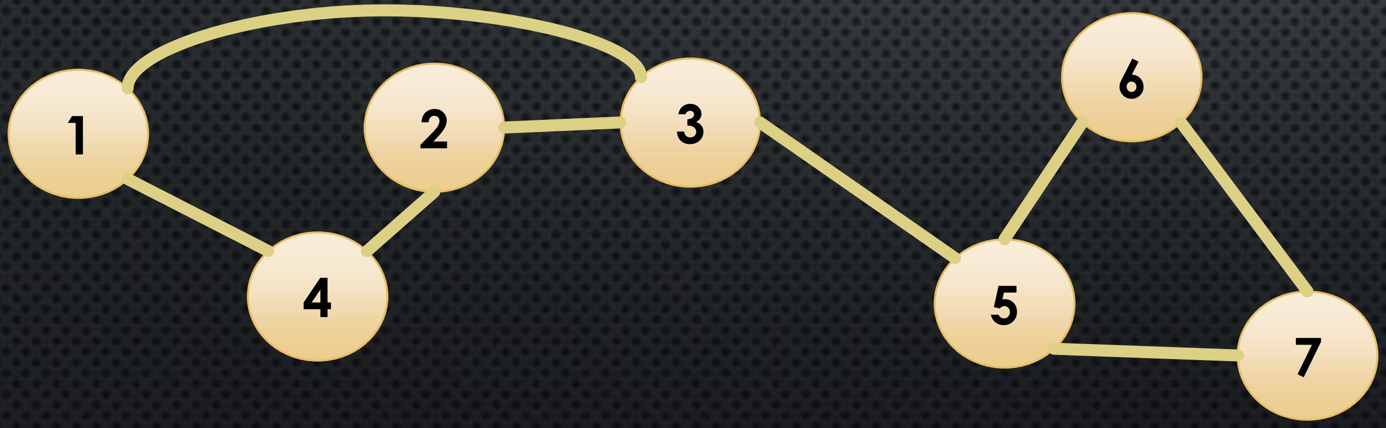
ADJACENCY LIST REPRESENTATION

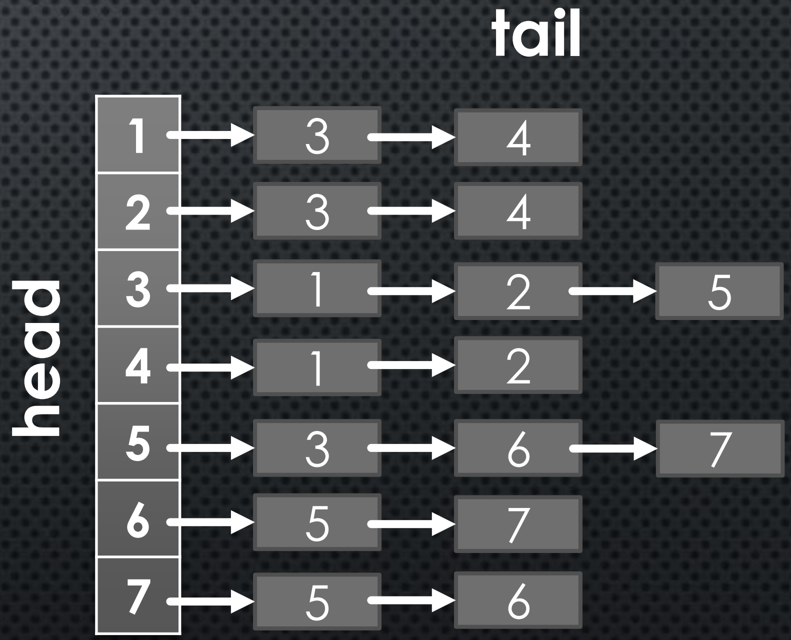
- n linked lists, one for each node
- We write $adj[u]$ to denote the list for node u
- $adj[u]$ contains the labels of nodes it has edges to



ADJACENCY LIST REPRESENTATION

- **For undirected graphs**
- If $adj[u]$ contains v then $adj[v]$ also contains u





1 → 3 → 4
2 → 3 → 4
3 → 1 → 2 → 5
4 → 1 → 2
5 → 3 → 6 → 7
6 → 5 → 7
7 → 5 → 6

IMPLEMENTING ADJACENCY LISTS

- Suppose we are loading a graph from input
    - Assume nodes are labeled 0..n-1
    - Array of lists adj[n]
    - (In C++, something like an array of vector<int> would work)

PROS AND CONS

	Adjacency matrix	**Adjacency list**
Time to test whether (u, v) is an edge	$O(1)$	$O(outdeg(u))$
Time to list neighbours of u	$O(n)$	$O(outdeg(u))$
Space complexity	$O(n^2)$	$O(n+m)$

Excellent when nodes have $O(1)$ **neighbours**

Can be better for dense graphs

Better if $o(n^2)$ edges

We call this a **sparse** graph

BREADTH FIRST SEARCH

A simple introduction to graph algorithms

```
BreadthFirstSearch(V[1..n], adj[1..n], s)
    pred[1..n] = [null, null, ..., null]
    dist[1..n] = [infty, infty, ..., infty]
    colour[1..n] = [white, white, ..., white]
    q = new queue

    colour[s] = gray
    dist[s] = 0
    q.enqueue(s)

    while q is not empty
        u = q.dequeue()
        for v in adj[u]
            if colour[v] = white
                pred[v] = u
                colour[v] = gray
                dist[v] = dist[u] + 1
                q.enqueue(v)
        colour[u] = black

    return colour, pred, dist
```

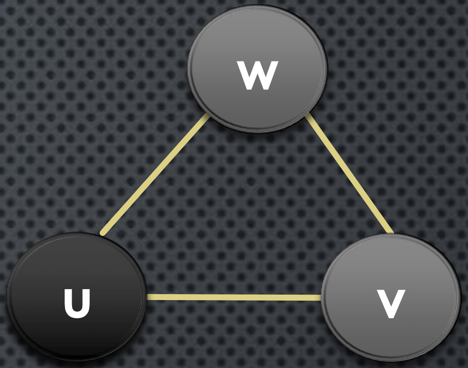
Assuming adjacency list representation

Discover (enqueue)
starting node s

Start processing
node u's edges

Discover
(enqueue)
neighbour v

Finish processing u



- **Undiscovered** nodes are **white**
- **Discovered** nodes are **gray**
  - **Processing** adjacent edges
- **Finished** nodes are **black**
  - Adjacent nodes have been **processed**
- **Connected graph:** each node is eventually black

```
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            if colour[v] = white
                pred[v] = u
                colour[v] = gray
                dist[v] = dist[u] + 1
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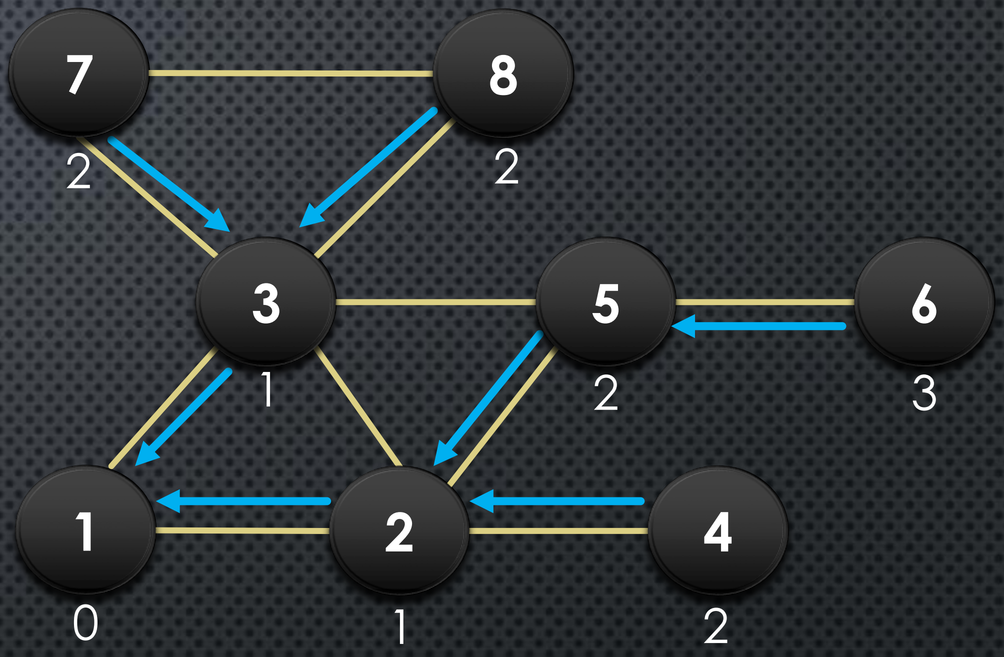
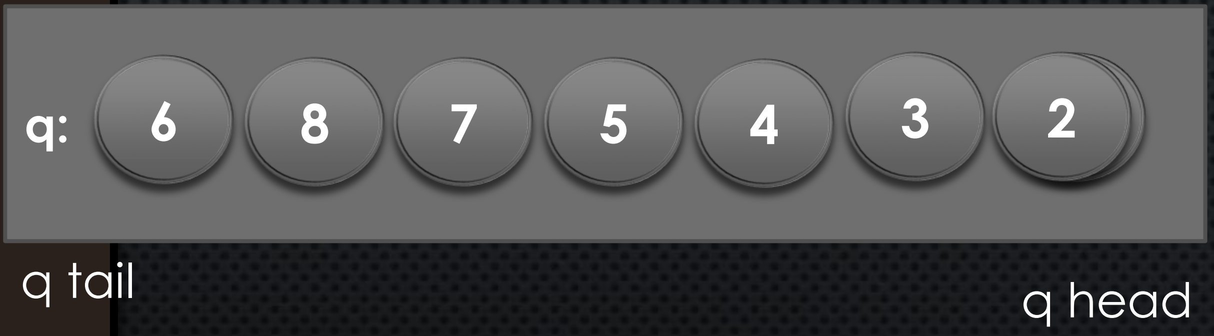
        colour[u] = black

    return colour, pred, dist
```

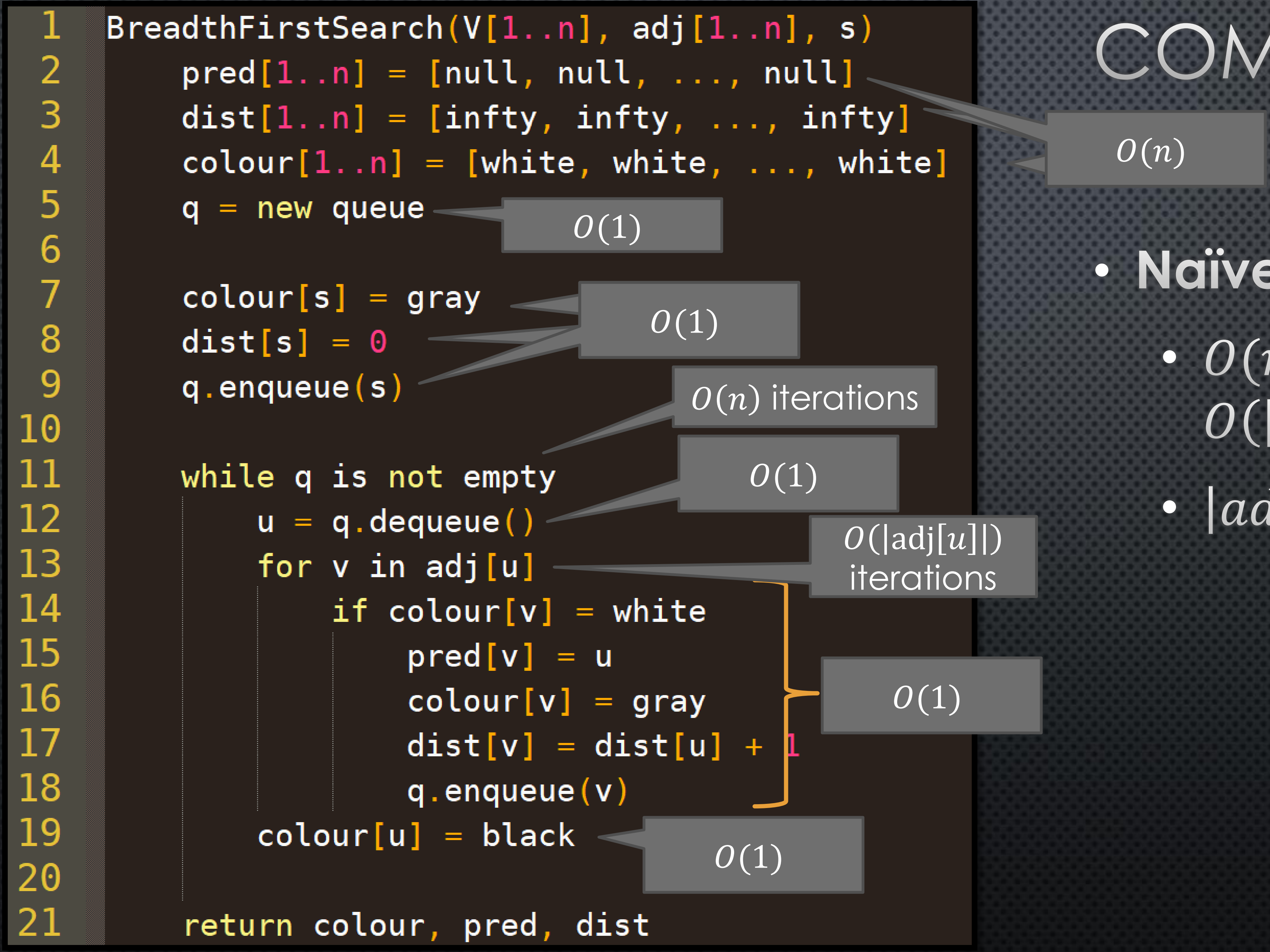
Example execution starting at node 1





COMPLEXITY

(with adjacency lists)



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    q = new queue

    colour[s] = gray
    dist[s] = 0
    q.enqueue(s)

    while q is not empty
        u = q.dequeue()
        for v in adj[u]
            if colour[v] = white
                pred[v] = u
                colour[v] = gray
                dist[v] = dist[u] + 1
                q.enqueue(v)
        colour[u] = black

    return colour, pred, dist
```

$O(n)$

$O(1)$

$O(1)$

$O(n)$ iterations

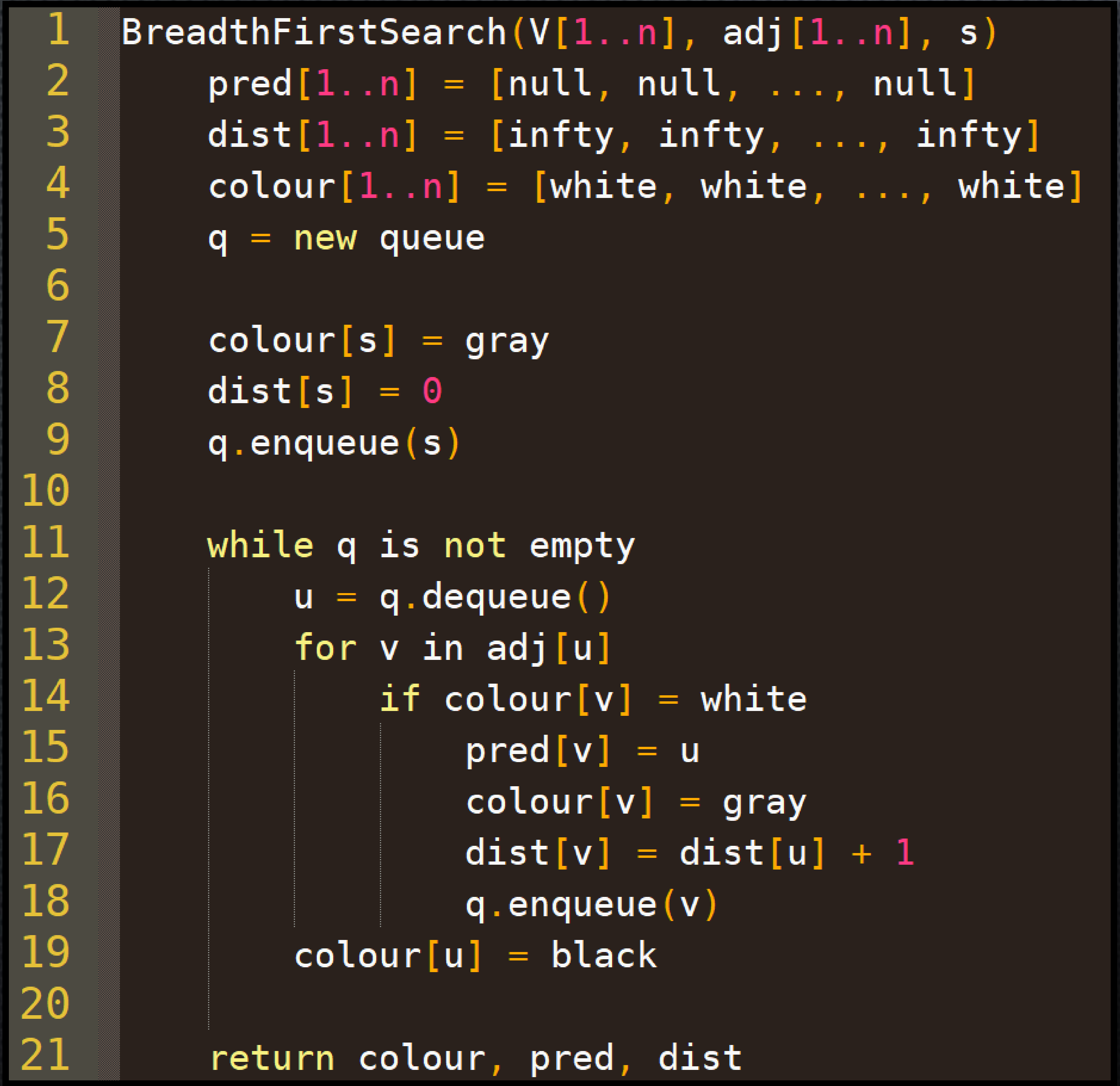
$O(1)$

$O(|\mathrm{adj}[u]|)$ iterations

$O(1)$

$O(1)$

- **Naïve loop analysis:**
  - $O(n)$ iterations * $O(|adj[u]|)$ iterations
  - $|adj[u]| \leq n$, so $O(n^2)$



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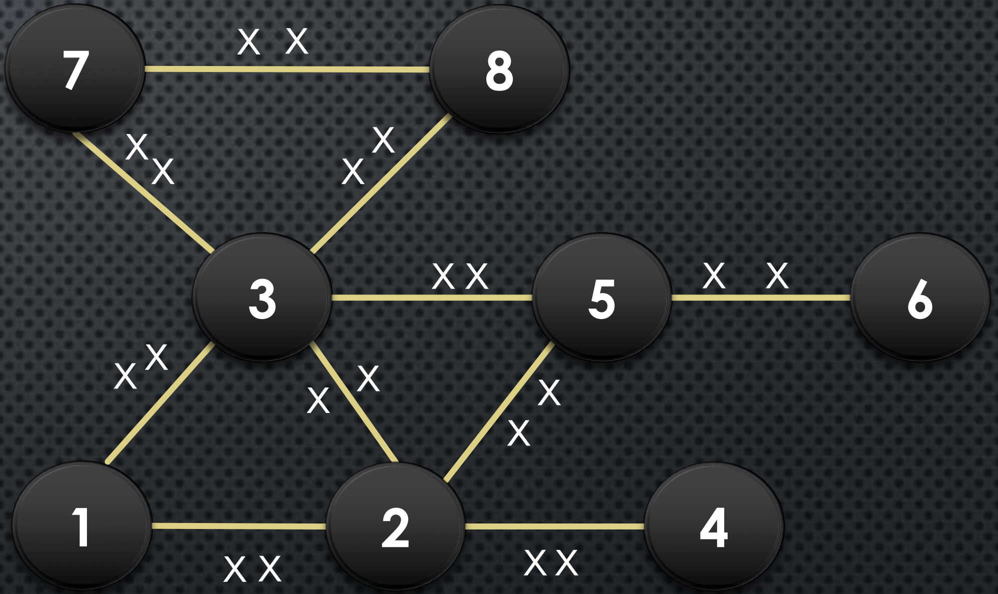
                dist[v] = dist[u] + 1
                q.enqueue(v)
        colour[u] = black

    return colour, pred, dist
```

Smarter loop analysis:

- For each u,
  iterate over all neighbours



- We touch each edge twice (doing $O(1)$ work each time)
- **Total contribution** of the inner loop to the runtime: $O(m)$

```
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    dist[s] = 0
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    while q is not empty
        u = q.dequeue()
        for v in adj[u]
            if colour[v] = white
                pred[v] = u
                colour[v] = gray
                dist[v] = dist[u] + 1
                q.enqueue(v)
        colour[u] = black

    return colour, pred, dist
```

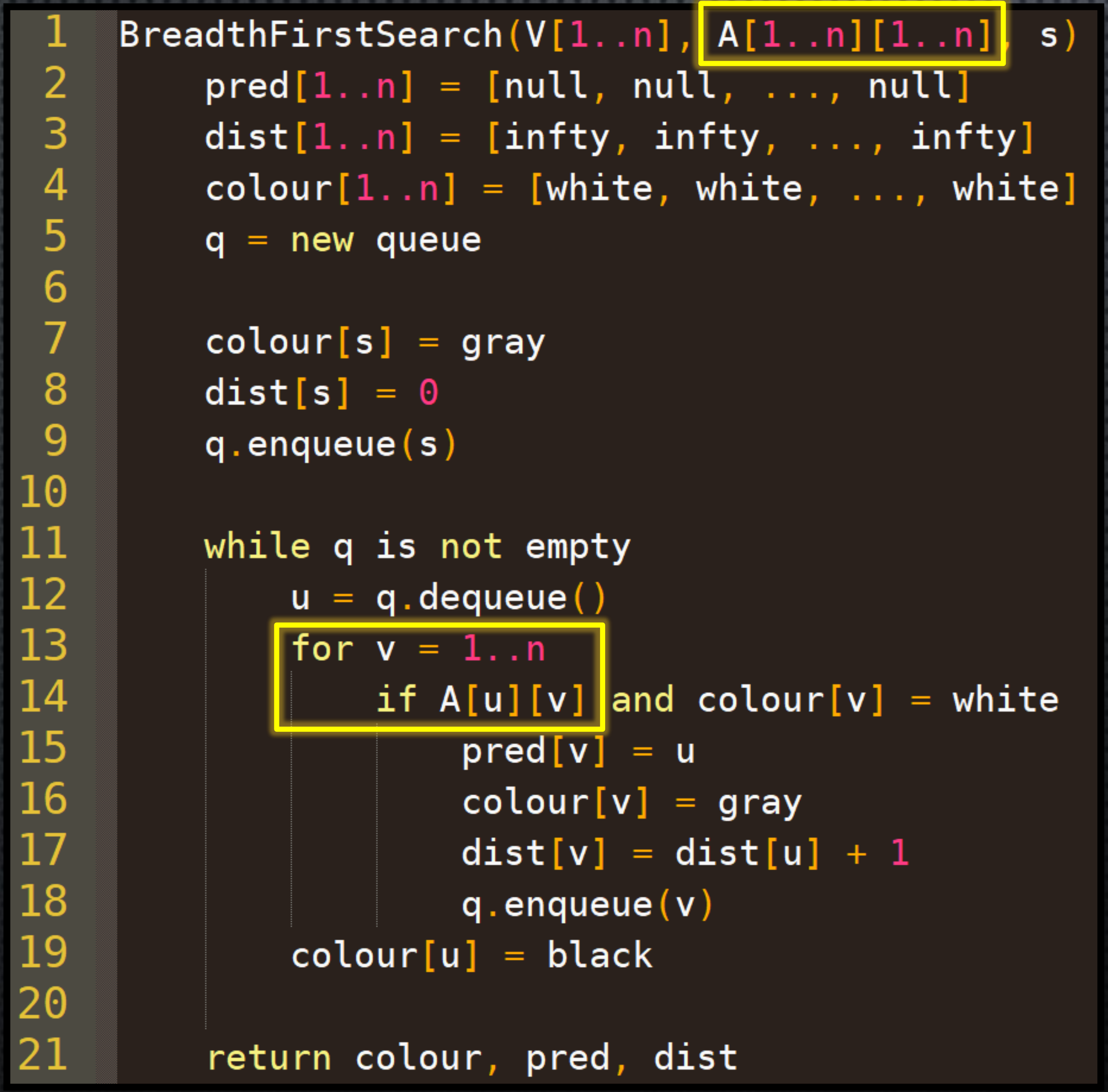
- **Smarter loop analysis:**
    - Initialization time: $\boldsymbol{O(n)}$
    - **Total contribution** of the **inner loop:** $\boldsymbol{O(m)}$
        - (**Over all iterations** of the outer loop)
    - Additional contribution of the **outer loop**: $\boldsymbol{O(n)}$
    - Total runtime: $\boldsymbol{O(m+n)}$

Analytic expression for loop complexity:

$$T_{LOOP}(n) \in O\left(\sum_{u=1}^{n}(1+\deg(u))\right)$$

$$= O\left(n + \sum_{u=1}^{n}\deg(u)\right) = \boldsymbol{O(n+m)}$$

DIFFERENCES WITH ADJACENCY MATRICES



```
BreadthFirstSearch(V[1..n], A[1..n][1..n], s)
    pred[1..n] = [null, null, ..., null]
    dist[1..n] = [infty, infty, ..., infty]
    colour[1..n] = [white, white, ..., white]
    q = new queue

    colour[s] = gray
    dist[s] = 0
    q.enqueue(s)

    while q is not empty
        u = q.dequeue()
        for v = 1..n
            if A[u][v] and colour[v] = white
                pred[v] = u
                colour[v] = gray
                dist[v] = dist[u] + 1
                q.enqueue(v)
        colour[u] = black

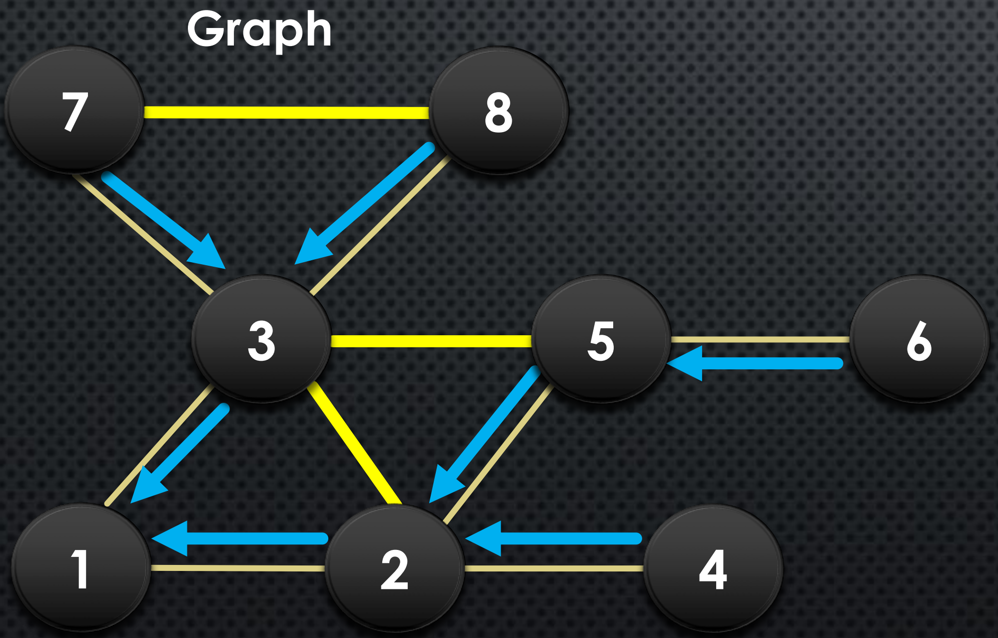
    return colour, pred, dist
```

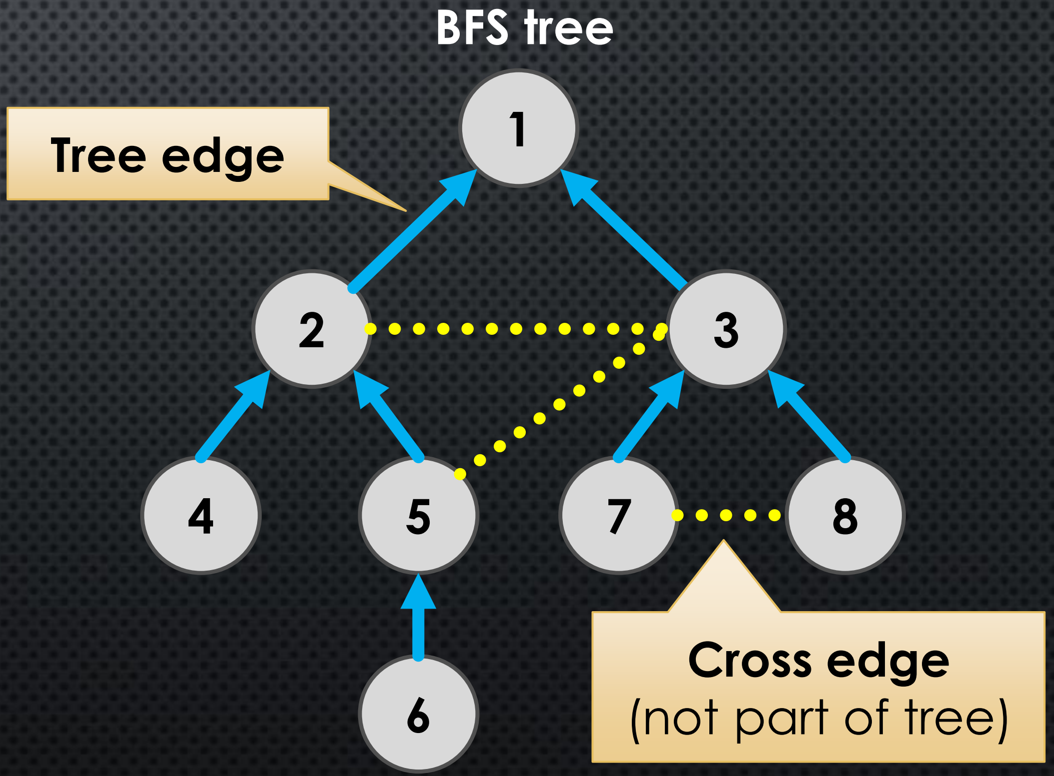
- Analysis is mostly similar
- **But**, it takes $O(n)$ time to determine which nodes are adjacent to u!
- This $O(n)$ cost is paid for each u, resulting in a **total runtime** $\in \boldsymbol{O(n^2)}$

BFS TREE

Disconnected? Forest…

- Connected graph: the $\boldsymbol{pred[]}$ array induces a **tree**
- The edges induced by $\boldsymbol{pred[]}$ are called **tree edges**
- Edges in the graph, but not in pred, are **cross edges**





Tree edge

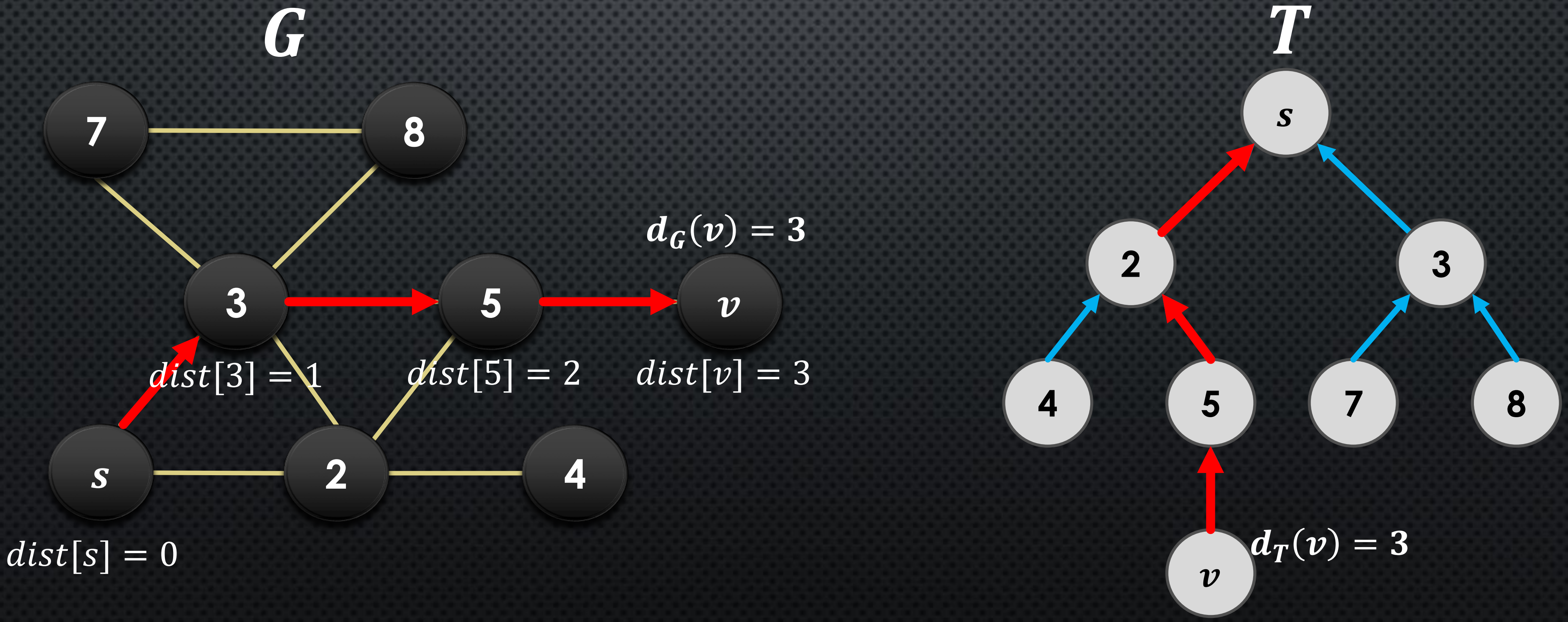
Cross edge
(not part of tree)

Careful: we will also see **DFS trees**, and cross edges will be defined differently

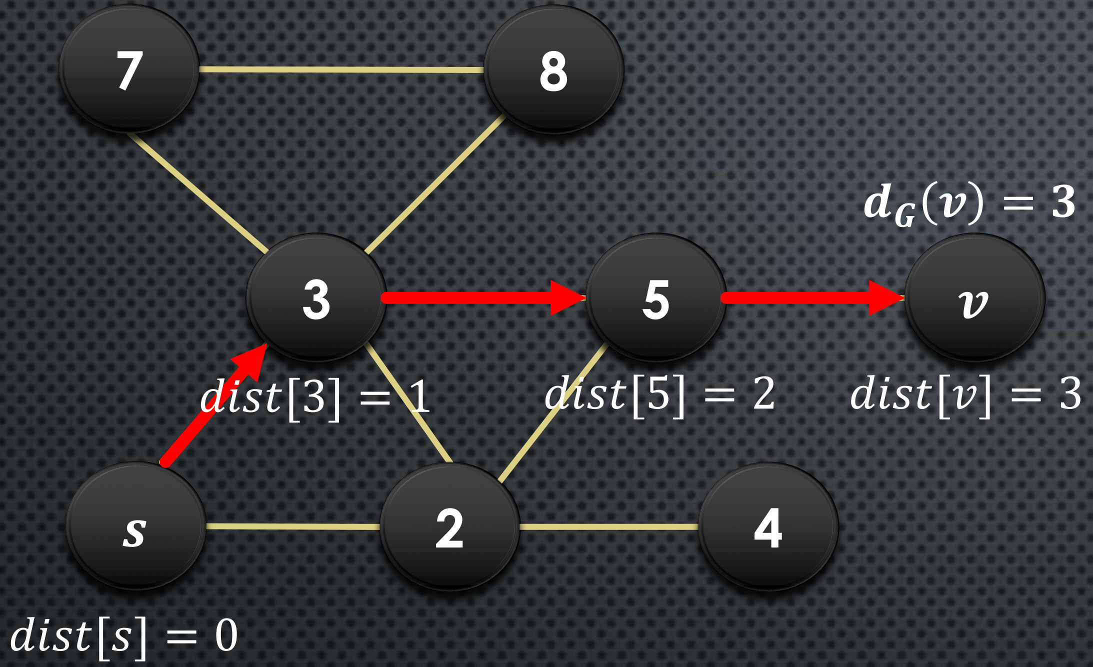
BFS: PROOF OF **OPTIMAL DISTANCES**

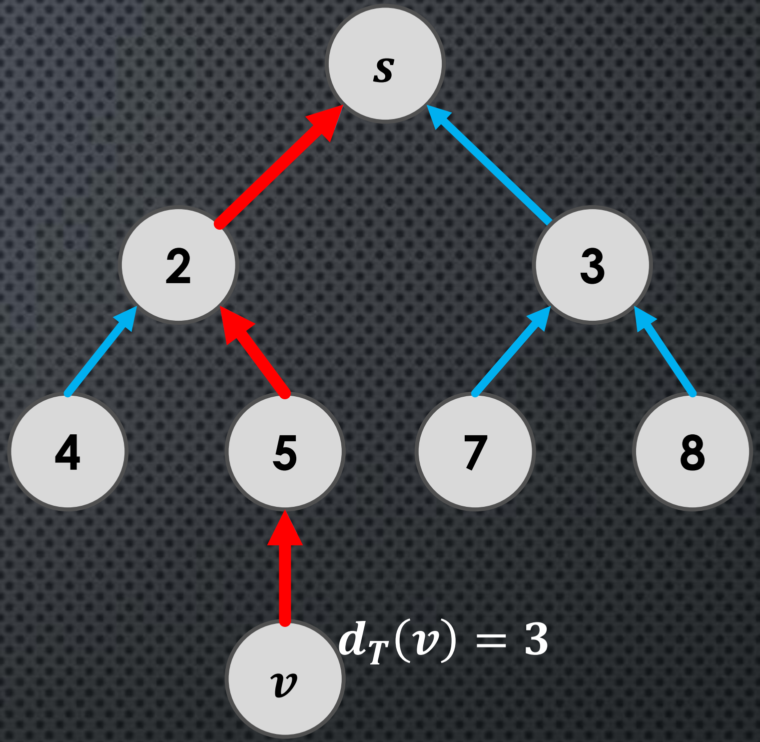
DISTANCE IN GRAPH G AND BFS TREE T

- Denote $d_G(v)$ as the (optimal) distance between s and v in G
- Denote $d_T(v)$ as the distance between s and v in the BFS tree T
- Recall: $dist[v]$ is a value set by BFS for each node v



PROOF IDEA

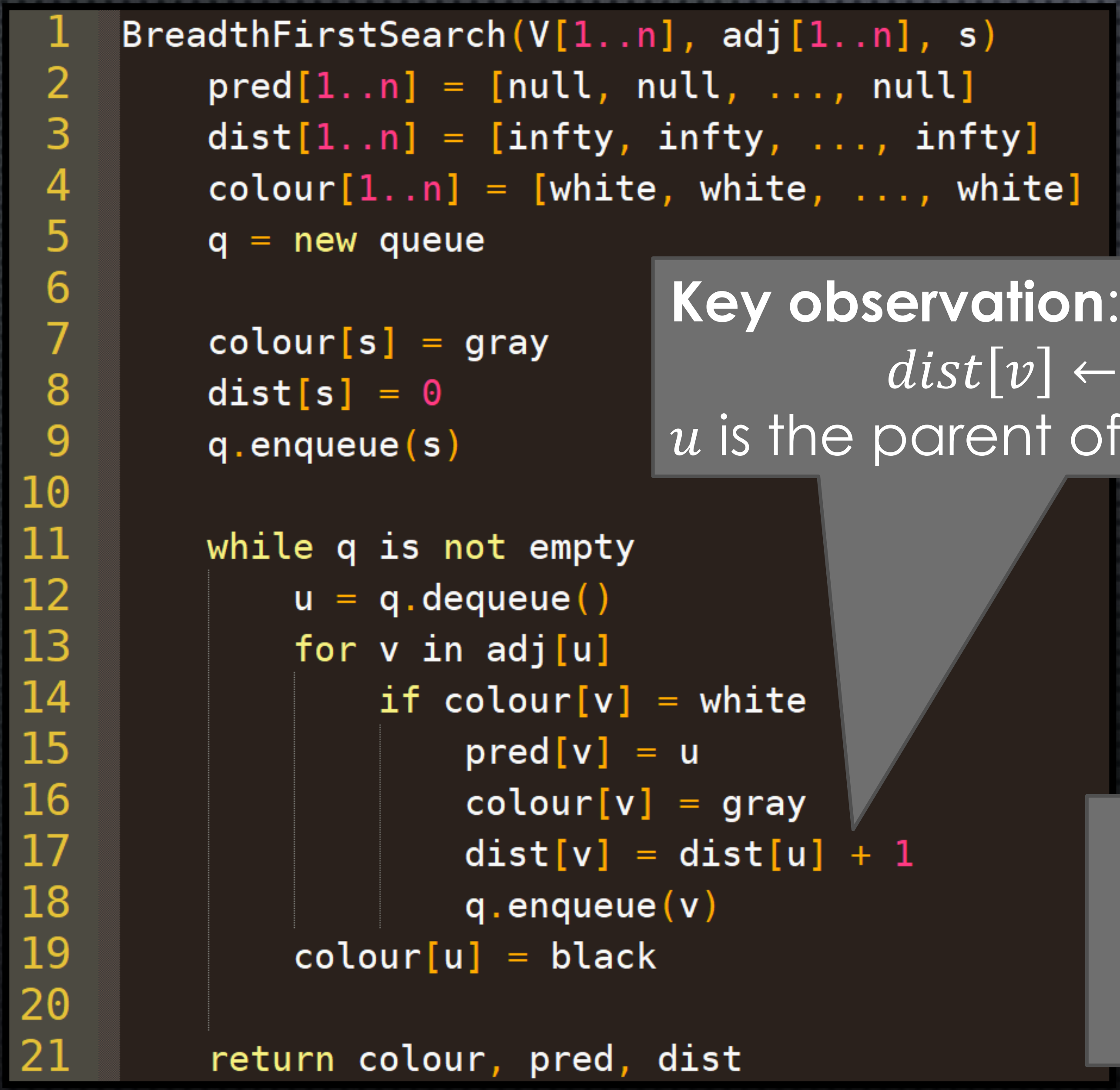




Want to show: at the end of BFS, $dist[v] = d_G(v)$ for all v

Plan: prove this in two parts
Claim 1: $dist[v] = d_T(v)$
Claim 2: $d_T(v) = d_G(v)$

SKETCH OF **CLAIM 1**: $dist[v] = d_T(v), \forall v \in V$



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                dist[v] = dist[u] + 1
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        colour[u] = black

    return colour, pred, dist
```

Key observation: whenever we set

$$dist[v] \leftarrow dist[u] + 1,$$

u is the parent of v in the BFS tree.

Based on this observation,
a simple inductive proof shows

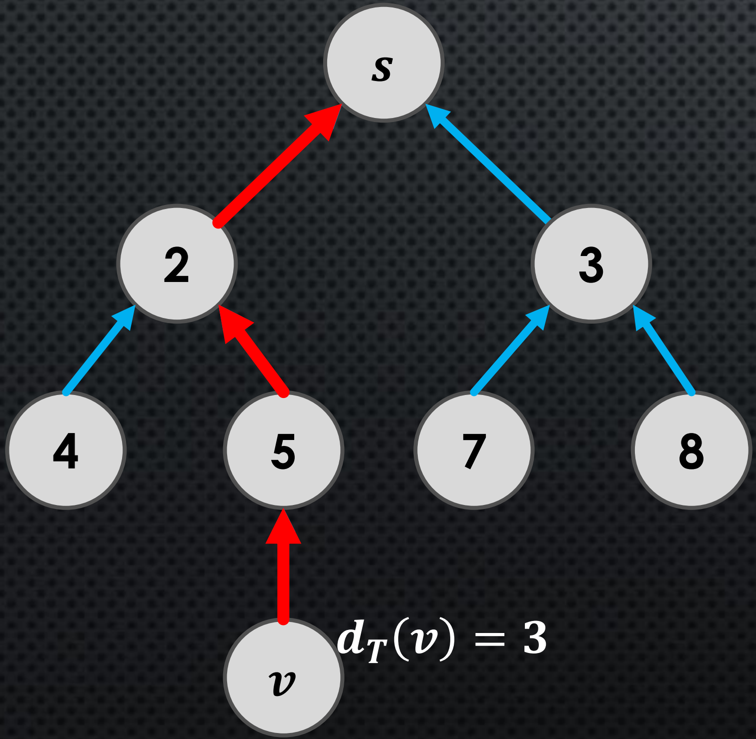
$$\boldsymbol{dist[v] = d_T(v)}$$

(for example, by strong induction on the nodes in the order their $dist$ values are set---left as an exercise)

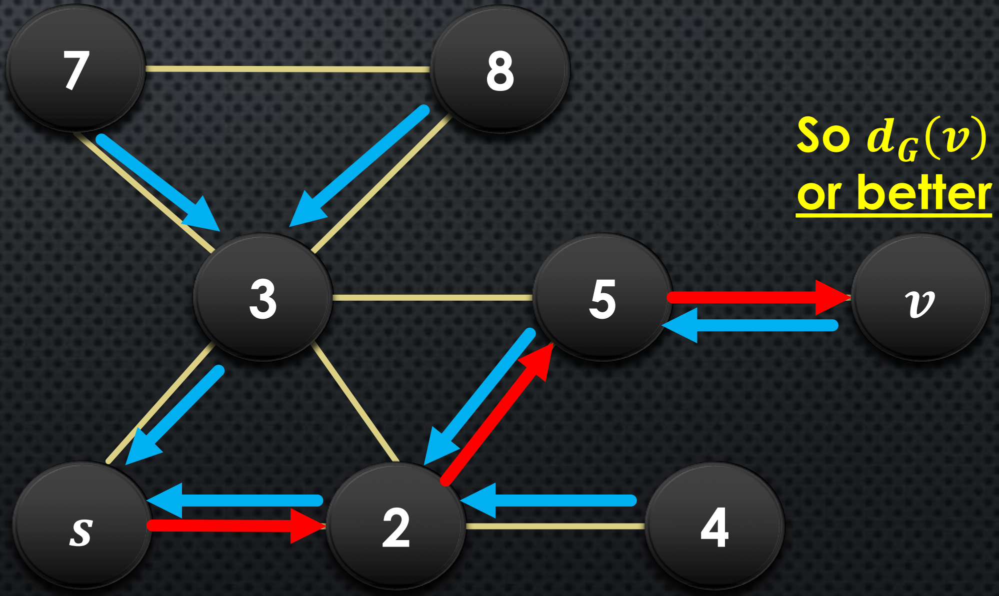
SKETCH OF **CLAIM 2**: $d_T(v) = d_G(v)$

To prove =
we show ≤ and ≥

- **Part 1:** $\forall v, \boldsymbol{d_G(v) \leq d_T(v)}$
  - There is a unique path $v \to \cdots \to s$ in T
  - And T is a **subgraph of G**
  - So that same path also exists in G (technically reversed)



$d_T(v) = 3$

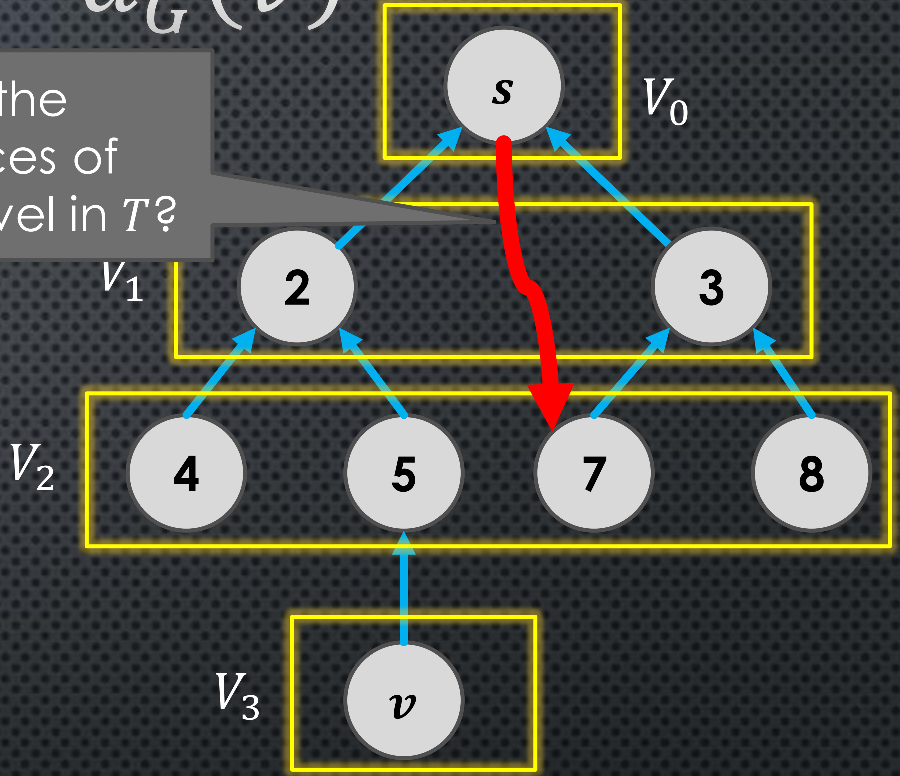


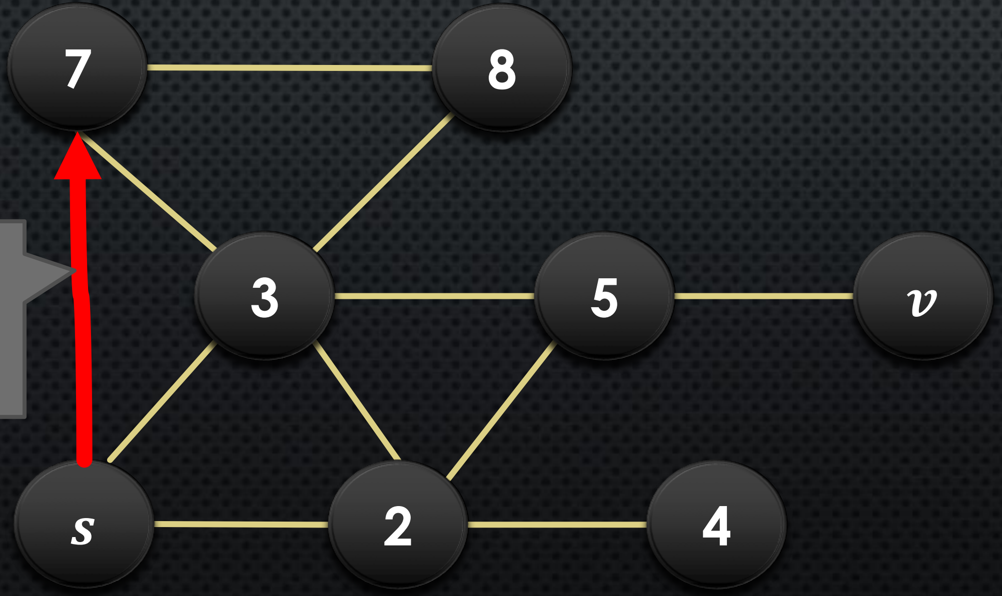
So $d_G(v)$ is 3 or better

SKETCH OF **CLAIM 2**: $d_T(v) = d_G(v)$

- **Part 2:** $\forall v, \boldsymbol{d_G(v) \geq d_T(v)}$
  - Partition T into **levels** $V_i = \{v: d_T(v) = i\}$ by distance from s
  - **Claim:** there is **no "forward" edge in G** that "skips" a level from V_i to $V_j, j \geq i + 2$
  - Suppose there is, for contradiction…

What are the consequences of "skipping" a level in T?





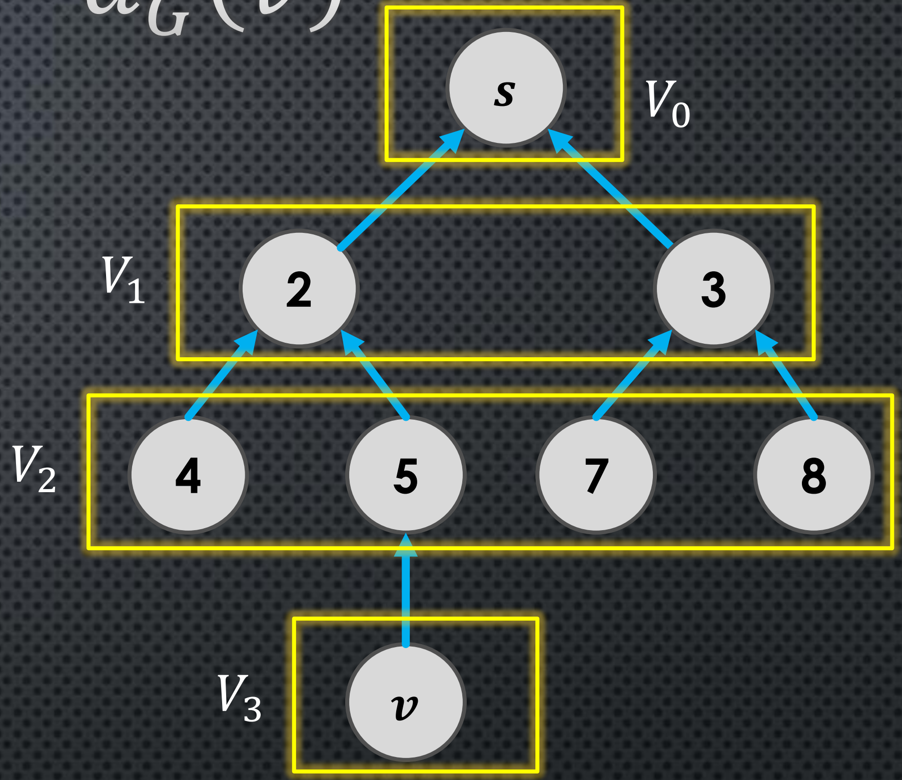
That "skip" edge in T looks like this in G

But that edge in G would cause 7 to have s as its parent, so $dist[7]$ would be **only 1 greater** than its parent…

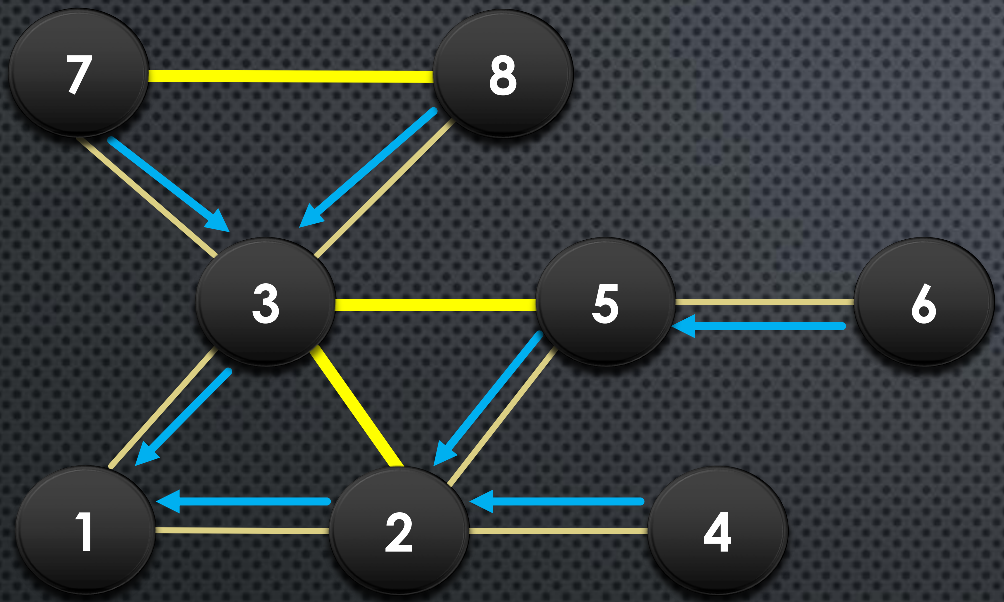
Contradicts(!) the assumption that the edge points to a node with **greater distance by at least 2**

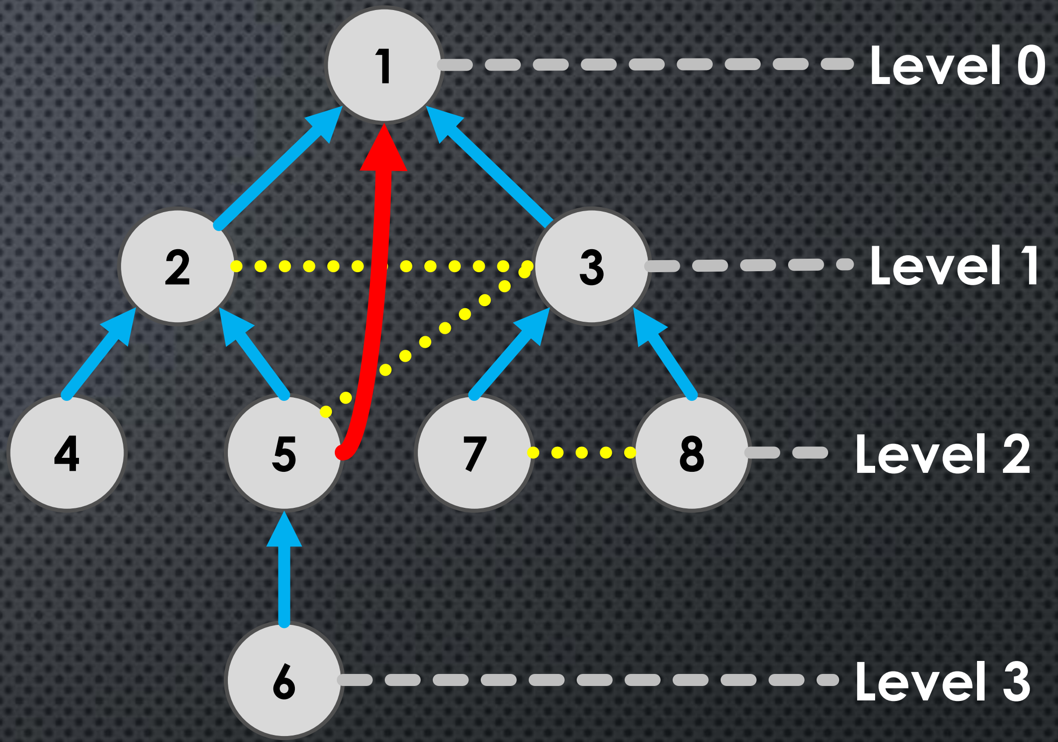
SKETCH OF **CLAIM 2**: $d_T(v) = d_G(v)$

- **Part 2:** $\forall v, \boldsymbol{d_G(v) \geq d_T(v)}$
  - We've just argued that there is **no "forward" edge in G** that "skips" a level in T from V_i to $V_j, j \geq i + 2$
  - Since no edge in G "skips" a level in T, we know **at least one edge in G** is needed to traverse **each level** between $\boldsymbol{s \in V_0}$ and $\boldsymbol{v \in V_{d_T(v)}}$
  - There are $d_{T(v)}$ such levels, so $d_G(v) \geq d_T(v)$



BFS TREE PROPERTIES





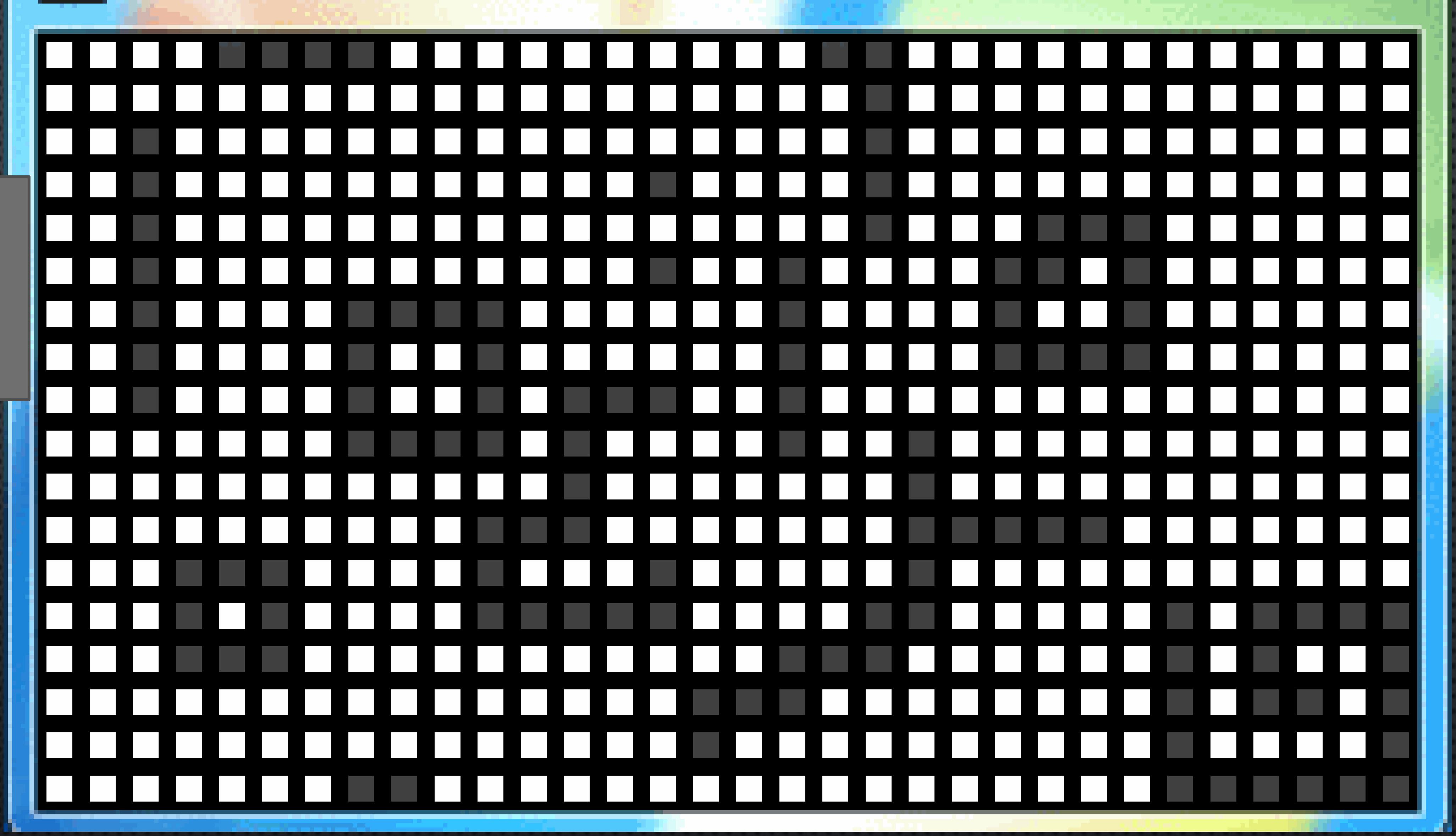
Fact: there are no "back" edges in <u>undirected</u> graphs that "skip" a level going <u>up</u> in the BFS tree.

Exercise: what about directed graphs?

Answer in bonus slides…

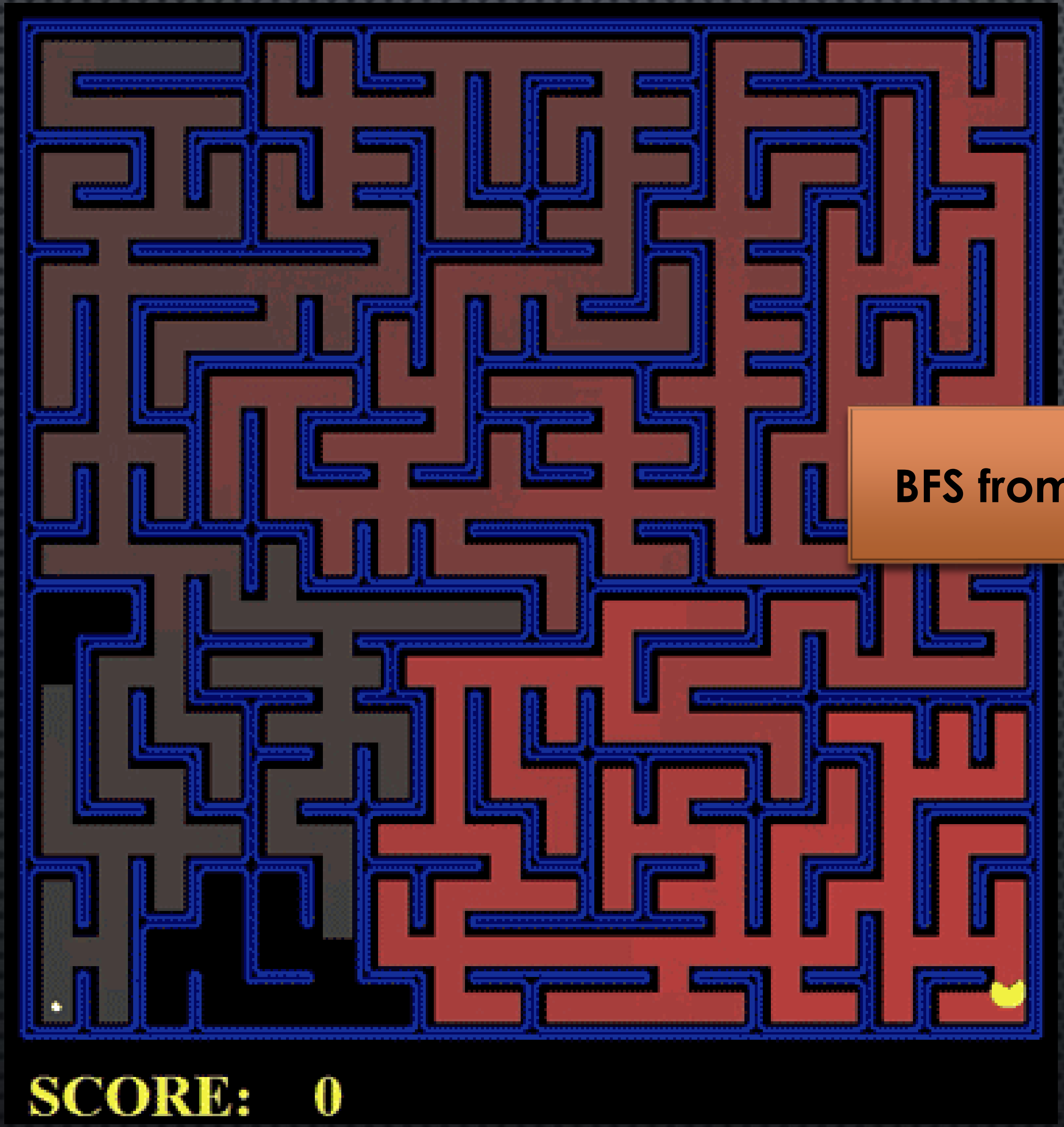
APPLICATION: FINDING **SHORTEST PATHS**

User interfaces: rubber-banding a **mouse cursor** around obstacles



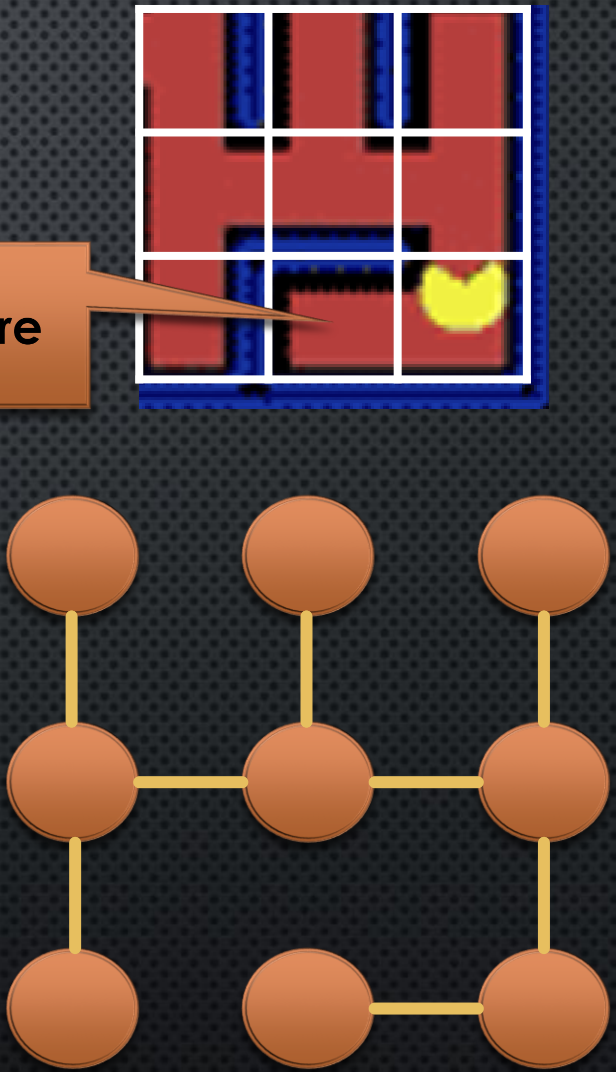
Starting to get into the details

Game AI:
path finding
in a **grid**-graph



BFS from here

How to **represent** a grid graph?



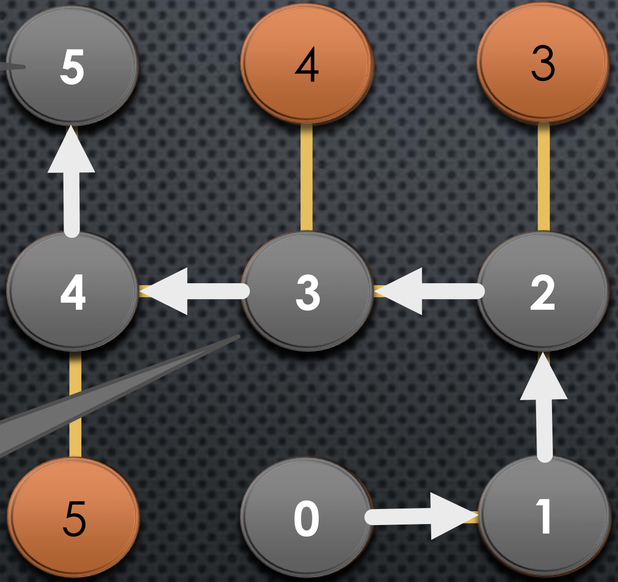
HOW TO **OUTPUT** AN **ACTUAL PATH**

- Suppose you want to output a **path** from s to v with minimum distance (not just the **distance** to v)
- Algorithm (what do you think?)
  - Similar to extracting an answer from a DP array!
  - Work backwards through the predecessors
  - Note: this will print the path **in reverse**! Solution?

Destination v

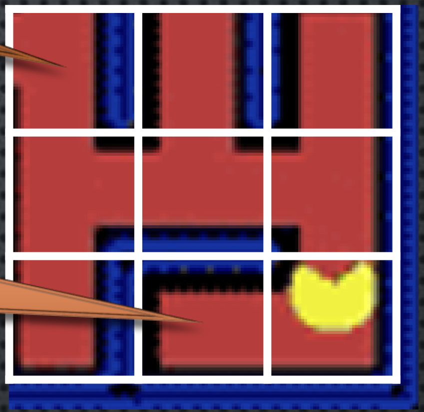
Predecessor $pred[v]$

Predecessor $pred[pred[v]]$



Shortest path **to here?**

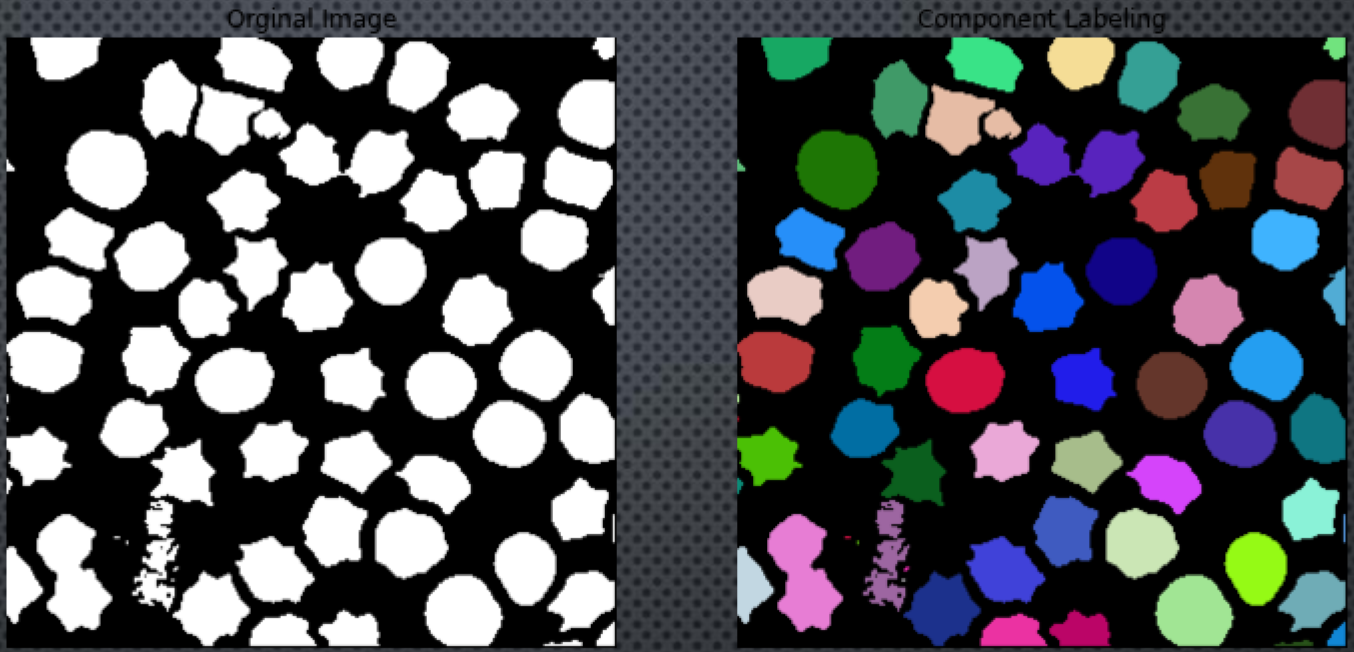
BFS from here



Each time you visit a predecessor, push it into a **stack**

I.e., push $v = 5$, then push $pred[v] = 4$, then push $pred[pred[v]] = 3$, then 2, …

At the end, **pop** all off the stack. This gives 0, 1, 2, …, 5 = **the path!**

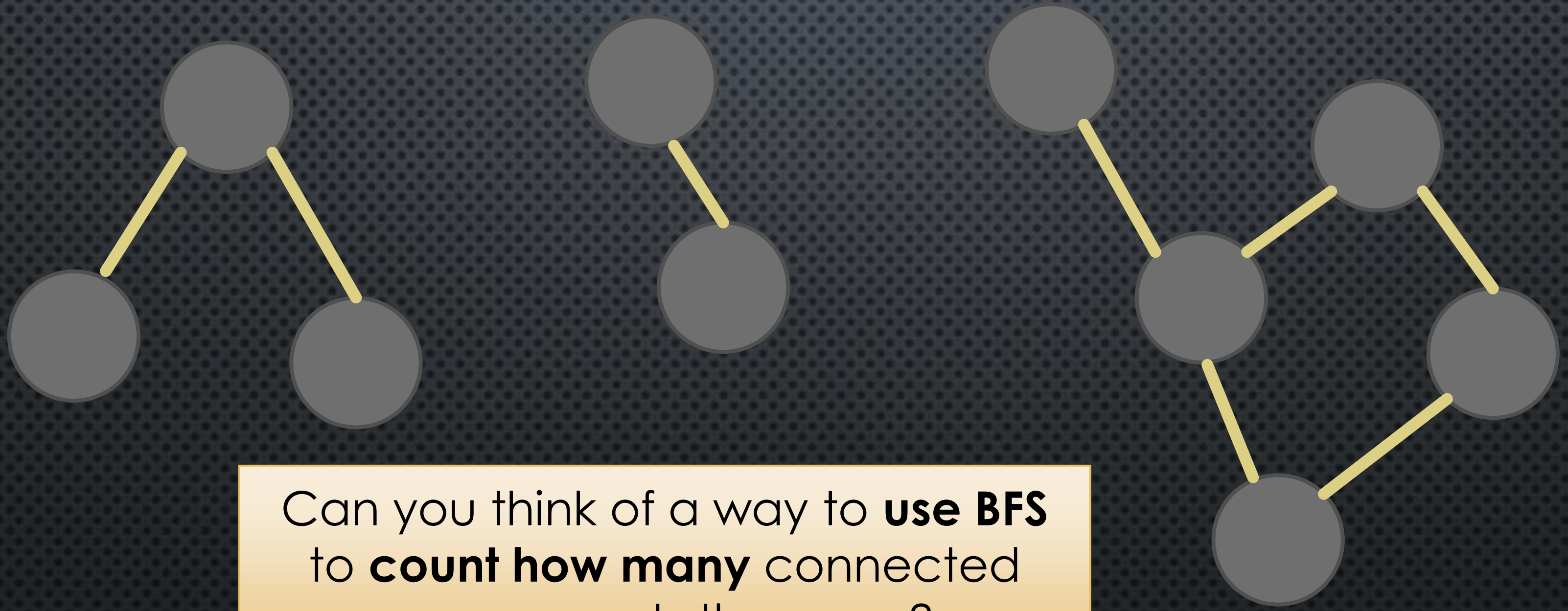


APPLICATION:

UNDIRECTED CONNECTED COMPONENTS

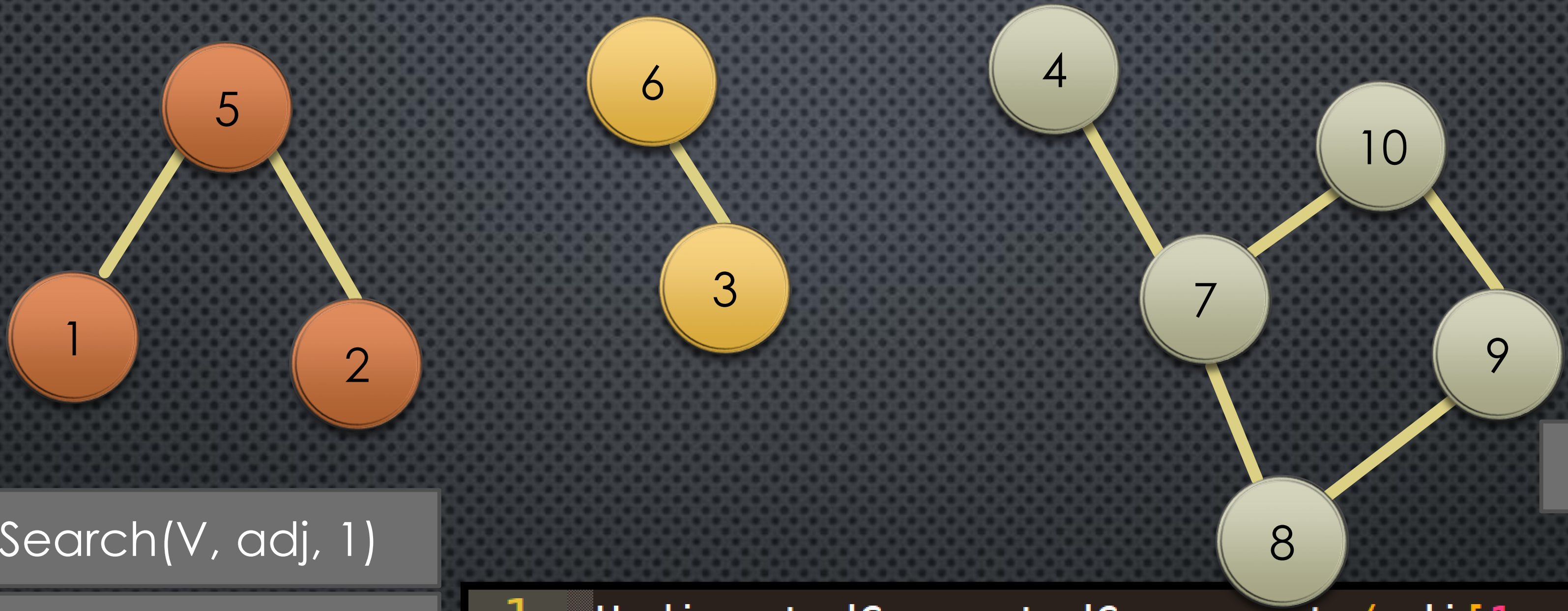
CONNECTED COMPONENTS

- Example: **undirected graph** with three **components**



Can you think of a way to **use BFS** to **count how many** connected components there are?

CONNECTED COMPONENTS



Can be done in $O(n + m)$ time

Complexity?

BreadthFirstSearch(V, adj, 1)

BreadthFirstSearch(V, adj, 3)

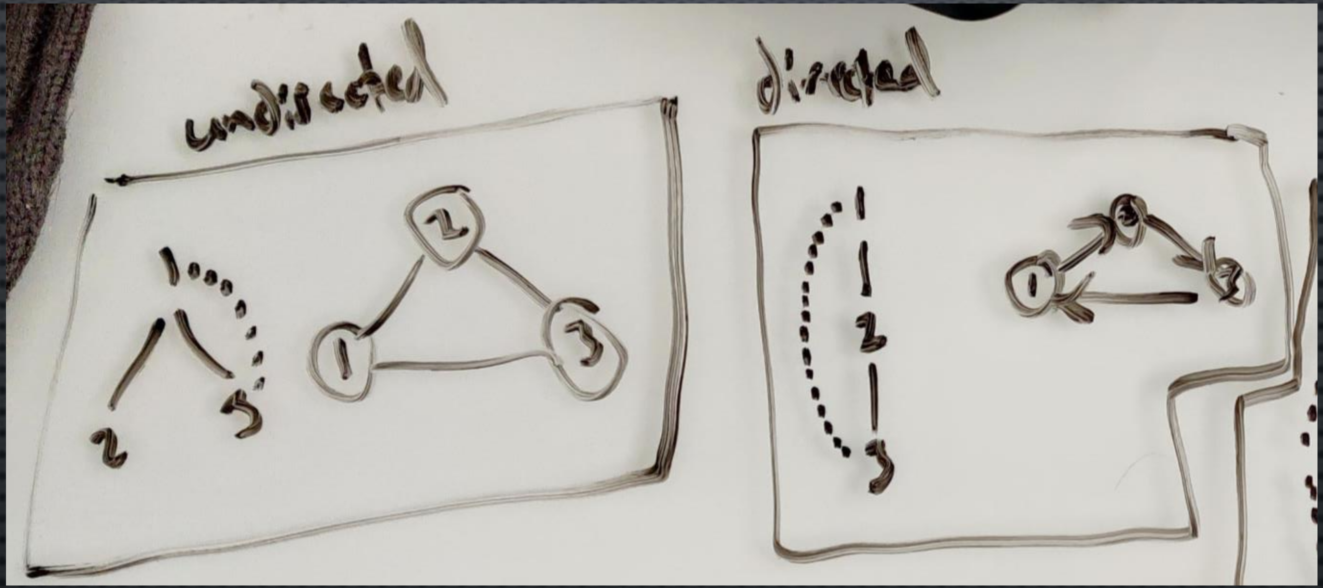
BreadthFirstSearch(V, adj, 4)

Modified BFS that (1) reuses the same colour array for consecutive calls and (2) sets comp[u] = compNum for each node u it visits

```
UndirectedConnectedComponents(adj[1..n])
    colour[1..n] = [white, ..., white]
    comp[1..n] = [0, ..., 0]
    compNum = 1
    for start = 1..n
        if colour[start] is white
            BFS(adj, start, colour, comp, compNum)
            compNum = compNum + 1
    return comp
```

BONUS SLIDES

ANSWER TO BFS TREE PROPERTY EXERCISE…



Bfs tree graph Bfs tree graph

Dotted = back edge