CS 341: ALGORITHMS

Lecture 11: graph algorithms II – finishing BFS, depth first search

Readings: see website

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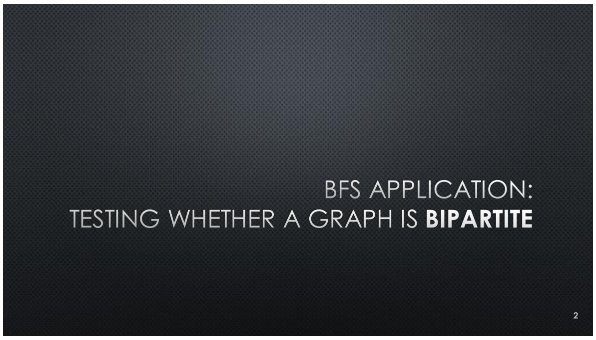
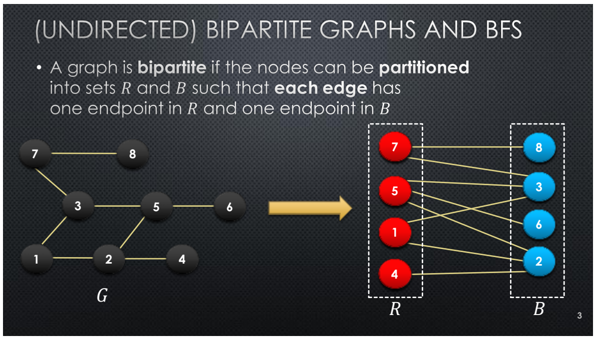
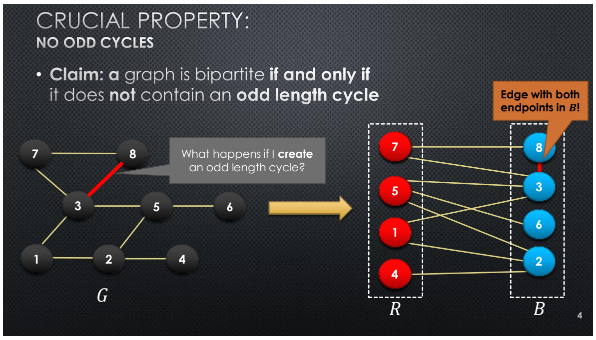


BFS APPLICATION:
TESTING WHETHER A GRAPH IS **BIPARTITE**



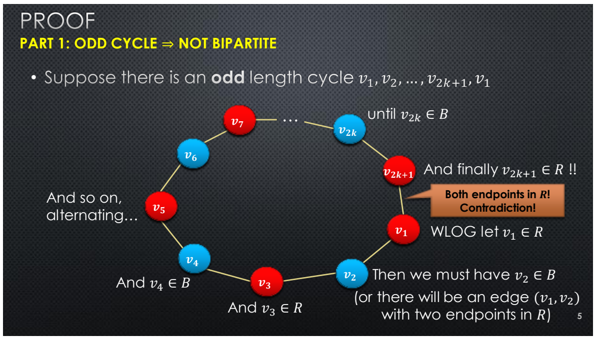
(UNDIRECTED) BIPARTITE GRAPHS AND BFS

- A graph is **bipartite** if the nodes can be **partitioned** into sets R and B such that **each edge** has one endpoint in R and one endpoint in B



CRUCIAL PROPERTY:
NO ODD CYCLES

- **Claim: a** graph is bipartite **if and only if** it does **not** contain an **odd length cycle**



PROOF
PART 1: ODD CYCLE ⇒ NOT BIPARTITE

- Suppose there is an **odd** length cycle $v_1, v_2, \dots, v_{2k+1}, v_1$

WLOG let $v_1 \in R$

Then we must have $v_2 \in B$
(or there will be an edge (v_1, v_2)
with two endpoints in R)

And $v_3 \in R$

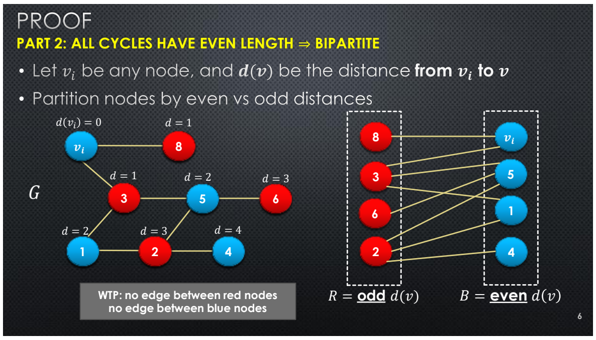
And $v_4 \in B$

And so on,
alternating…

until $v_{2k} \in B$

And finally $v_{2k+1} \in R$!!

Both endpoints in R! Contradiction!

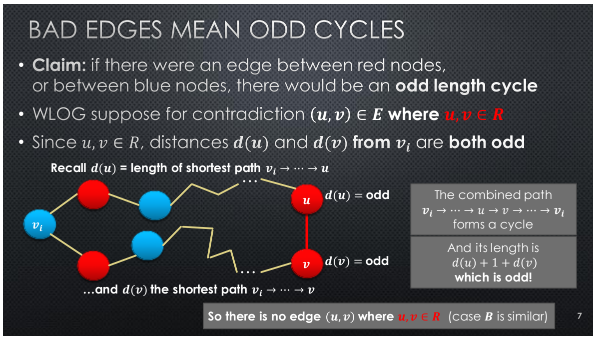


PROOF
PART 2: ALL CYCLES HAVE EVEN LENGTH ⇒ BIPARTITE

- Let v_i be any node, and $\boldsymbol{d(v)}$ be the distance **from v_i to v**
- Partition nodes by even vs odd distances

**WTP: no edge between red nodes
no edge between blue nodes**

$R =$ **<u>odd</u>** $d(v)$ $B =$ **<u>even</u>** $d(v)$



BAD EDGES MEAN ODD CYCLES

- **Claim:** if there were an edge between red nodes, or between blue nodes, there would be an **odd length cycle**
- WLOG suppose for contradiction $(u, v) \in E$ **where** $u, v \in R$
- Since $u, v \in R$, distances $d(u)$ and $d(v)$ **from** v_i are **both odd**

Recall $d(u)$ **= length of shortest path** $v_i \rightarrow \cdots \rightarrow u$

$d(u) = $ **odd**

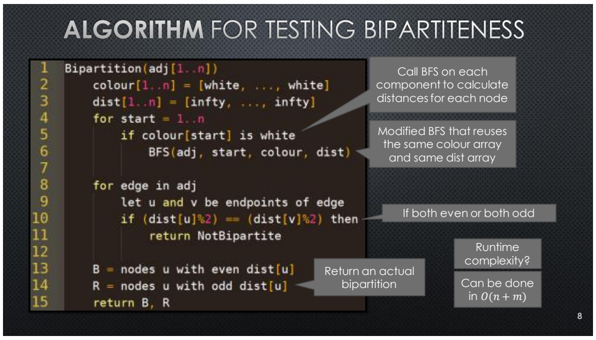
$d(v) = $ **odd**

…and $d(v)$ **the shortest path** $v_i \rightarrow \cdots \rightarrow v$

The combined path $v_i \rightarrow \cdots \rightarrow u \rightarrow v \rightarrow \cdots \rightarrow v_i$ forms a cycle

And its length is $d(u) + 1 + d(v)$ **which is odd!**

So there is no edge (u, v) **where** $u, v \in R$ (case B is similar)



ALGORITHM FOR TESTING BIPARTITENESS

```
Bipartition(adj[1..n])
    colour[1..n] = [white, ..., white]
    dist[1..n] = [infty, ..., infty]
    for start = 1..n
        if colour[start] is white
            BFS(adj, start, colour, dist)

    for edge in adj
        let u and v be endpoints of edge
        if (dist[u]%2) == (dist[v]%2) then
            return NotBipartite

    B = nodes u with even dist[u]
    R = nodes u with odd dist[u]
    return B, R
```

Call BFS on each component to calculate distances for each node

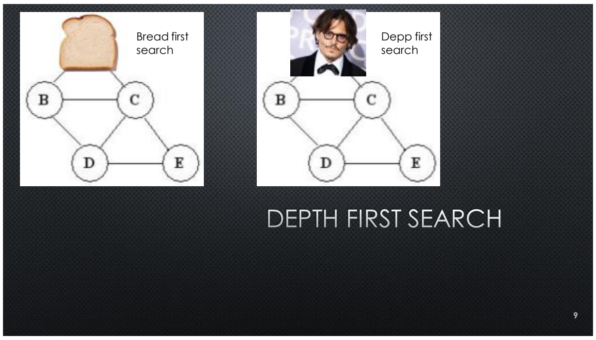
Modified BFS that reuses the same colour array and same dist array

If both even or both odd

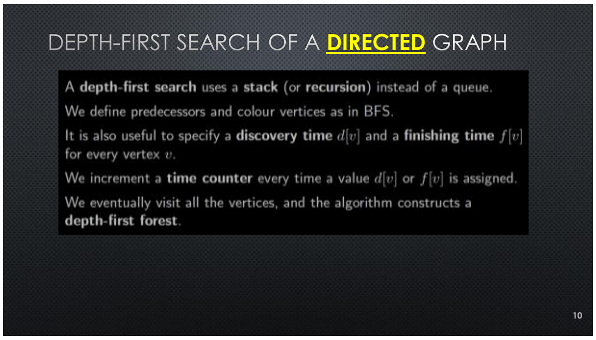
Return an actual bipartition

Runtime complexity?

Can be done in $O(n + m)$



DEPTH FIRST SEARCH



DEPTH-FIRST SEARCH OF A DIRECTED GRAPH

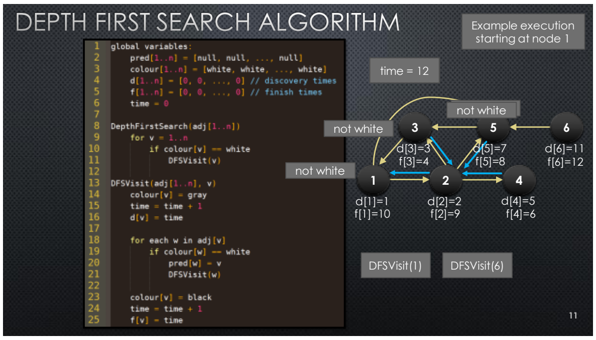
A **depth-first search** uses a **stack** (or **recursion**) instead of a queue.

We define predecessors and colour vertices as in BFS.

It is also useful to specify a **discovery time** $d[v]$ and a **finishing time** $f[v]$ for every vertex v.

We increment a **time counter** every time a value $d[v]$ or $f[v]$ is assigned.

We eventually visit all the vertices, and the algorithm constructs a **depth-first forest**.



DEPTH FIRST SEARCH ALGORITHM

```
global variables:
    pred[1..n] = [null, null, ..., null]
    colour[1..n] = [white, white, ..., white]
    d[1..n] = [0, 0, ..., 0] // discovery times
    f[1..n] = [0, 0, ..., 0] // finish times
    time = 0

DepthFirstSearch(adj[1..n])
    for v = 1..n
        if colour[v] == white
            DFSVisit(v)

DFSVisit(adj[1..n], v)
    colour[v] = gray
    time = time + 1
    d[v] = time

    for each w in adj[v]
        if colour[w] == white
            pred[w] = v
            DFSVisit(w)

    colour[v] = black
    time = time + 1
    f[v] = time
```

Example execution starting at node 1

time = 12

not white

not white

not white

d[1]=1
f[1]=10

d[2]=2
f[2]=9

d[3]=3
f[3]=4

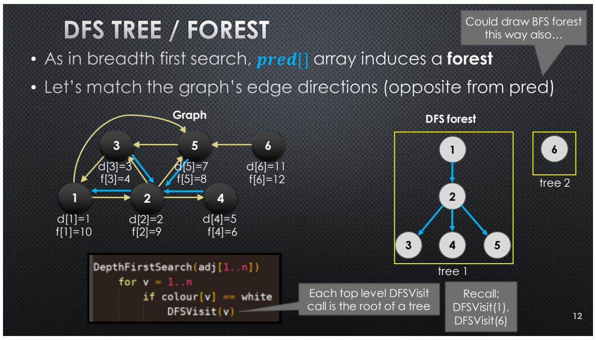
d[4]=5
f[4]=6

d[5]=7
f[5]=8

d[6]=11
f[6]=12

DFSVisit(1)

DFSVisit(6)



DFS TREE / FOREST

- As in breadth first search, $pred[]$ array induces a **forest**
- Let's match the graph's edge directions (opposite from pred)

Could draw BFS forest this way also…

Graph

d[1]=1
f[1]=10

d[2]=2
f[2]=9

d[3]=3
f[3]=4

d[4]=5
f[4]=6

d[5]=7
f[5]=8

d[6]=11
f[6]=12

DFS forest

tree 1

tree 2

```
DepthFirstSearch(adj[1..n])
    for v = 1..n
        if colour[v] == white
            DFSVisit(v)
```

Each top level DFSVisit call is the root of a tree

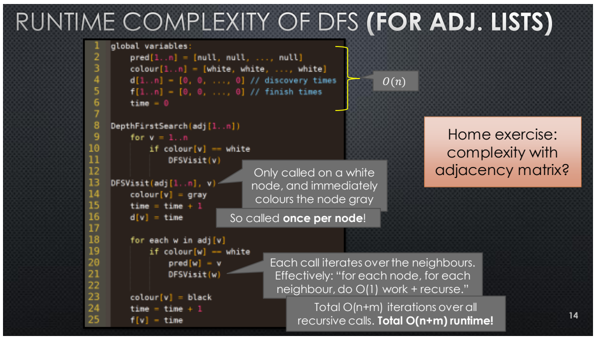
Recall: DFSVisit(1), DFSVisit(6)

BASIC DFS PROPERTIES TO REMEMBER

- Nodes start **white**
- A node v turns **gray** when it is **discovered**, which is when the first call to $DFSVisit(v)$ happens
- **After** v is turned **gray**, we recurse on its neighbours
- After recursing on **all neighbours**, we turn v **black**
  - Recursive calls on neighbours end before $DFSVisit(v)$ does, so the neighbours of v turn black before v

Also gets a **discovery time** $d[v]$ at this point

Also gets a **finish time** $f[v]$ at this point



RUNTIME COMPLEXITY OF DFS **(FOR ADJ. LISTS)**

```
global variables:
    pred[1..n] = [null, null, ..., null]
    colour[1..n] = [white, white, ..., white]
    d[1..n] = [0, 0, ..., 0] // discovery times
    f[1..n] = [0, 0, ..., 0] // finish times
    time = 0

DepthFirstSearch(adj[1..n])
    for v = 1..n
        if colour[v] == white
            DFSVisit(v)

DFSVisit(adj[1..n], v)
    colour[v] = gray
    time = time + 1
    d[v] = time

    for each w in adj[v]
        if colour[w] == white
            pred[w] = v
            DFSVisit(w)

    colour[v] = black
    time = time + 1
    f[v] = time
```

$O(n)$

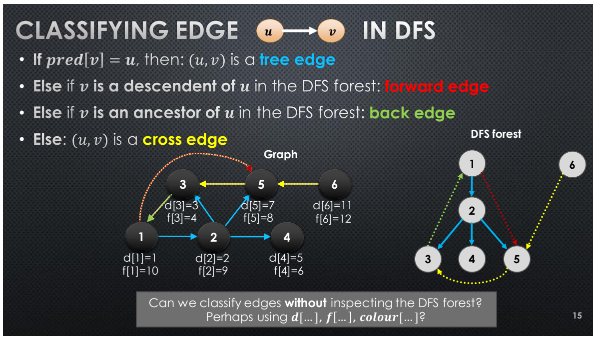
Only called on a white node, and immediately colours the node gray

So called **once per node**!

Each call iterates over the neighbours. Effectively: "for each node, for each neighbour, do O(1) work + recurse."

Total O(n+m) iterations over all recursive calls. **Total O(n+m) runtime!**

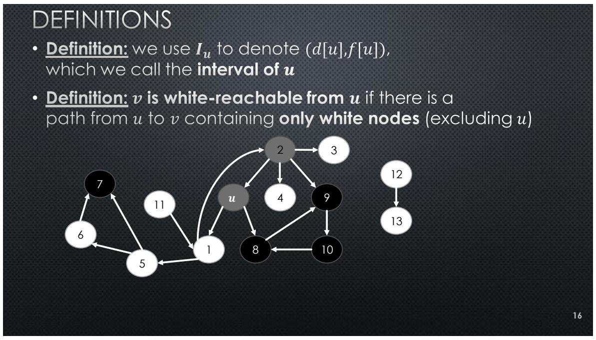
Home exercise: complexity with adjacency matrix?



CLASSIFYING EDGE $u \rightarrow v$ IN DFS

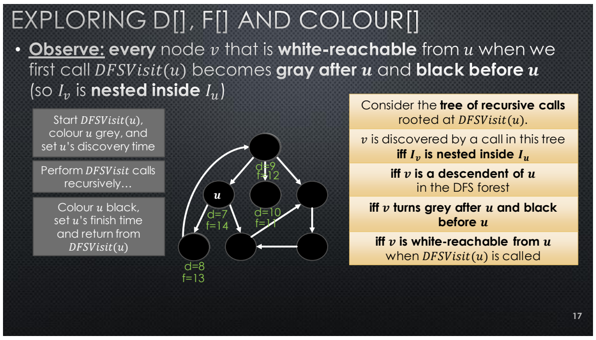
- **If** $pred[v] = u$, then: (u, v) is a **tree edge**
- **Else** if **v is a descendent of u** in the DFS forest: **forward edge**
- **Else** if **v is an ancestor of u** in the DFS forest: **back edge**
- **Else**: (u, v) is a **cross edge**

Can we classify edges **without** inspecting the DFS forest? Perhaps using $d[\ldots]$, $f[\ldots]$, $colour[\ldots]$?



DEFINITIONS

- **Definition:** we use I_u to denote $(d[u], f[u])$, which we call the **interval of u**
- **Definition:** **v is white-reachable from u** if there is a path from u to v containing **only white nodes** (excluding u)



EXPLORING D[], F[] AND COLOUR[]

- **Observe:** **every** node v that is **white-reachable** from u when we first call $DFSVisit(u)$ becomes **gray after u** and **black before u** (so I_v is **nested inside** I_u)

Start $DFSVisit(u)$, colour u grey, and set u's discovery time

Perform $DFSVisit$ calls recursively…

Colour u black, set u's finish time and return from $DFSVisit(u)$

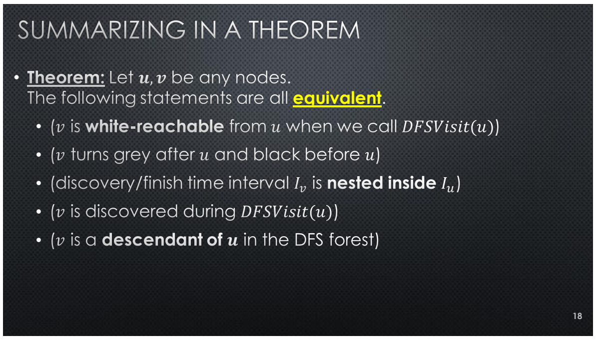
Consider the **tree of recursive calls** rooted at $DFSVisit(u)$.

v is discovered by a call in this tree **iff I_v is nested inside I_u**

iff v is a descendent of u in the DFS forest

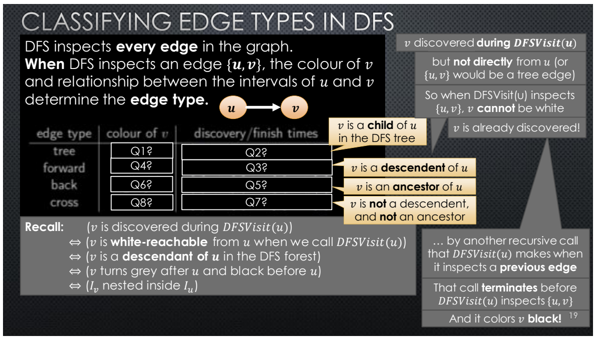
iff v turns grey after u and black before u

iff v is white-reachable from u when $DFSVisit(u)$ is called



SUMMARIZING IN A THEOREM

- **Theorem:** Let u, v be any nodes. The following statements are all **equivalent**.
  - (v is **white-reachable** from u when we call $DFSVisit(u)$)
  - (v turns grey after u and black before u)
  - (discovery/finish time interval I_v is **nested inside** I_u)
  - (v is discovered during $DFSVisit(u)$)
  - (v is a **descendant of u** in the DFS forest)



CLASSIFYING EDGE TYPES IN DFS

DFS inspects **every edge** in the graph.
When DFS inspects an edge $\{\boldsymbol{u},\boldsymbol{v}\}$, the colour of v and relationship between the intervals of u and v determine the **edge type.**

edge type	colour of v	discovery/finish times
tree	Q1?	Q2?
forward	Q4?	Q3?
back	Q6?	Q5?
cross	Q8?	Q7?

v is a **child** of u in the DFS tree

v is a **descendent** of u

v is an **ancestor** of u

v is **not** a descendent, and **not** an ancestor

Recall: (v is discovered during $DFSVisit(u)$)
⇔ (v is **white-reachable** from u when we call $DFSVisit(u)$)
⇔ (v is a **descendant of $\boldsymbol{u}$** in the DFS forest)
⇔ (v turns grey after u and black before u)
⇔ (I_v nested inside I_u)

v discovered **during $\boldsymbol{DFSVisit(u)}$**

but **not directly** from u (or $\{u,v\}$ would be a tree edge)

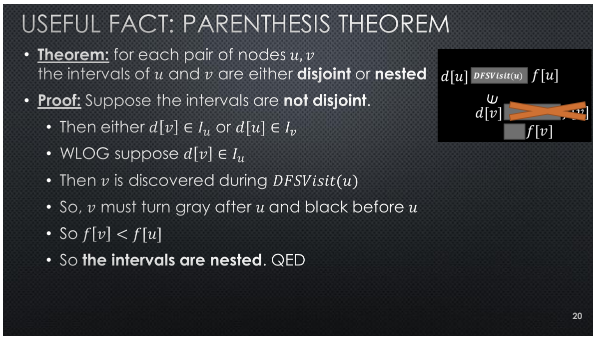
So when DFSVisit(u) inspects $\{u,v\}$, v **cannot** be white

v is already discovered!

… by another recursive call that $DFSVisit(u)$ makes when it inspects a **previous edge**

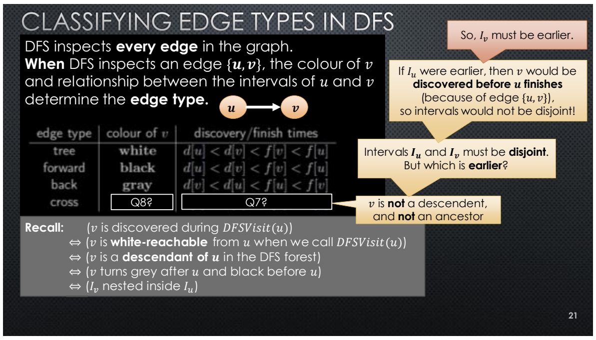
That call **terminates** before $DFSVisit(u)$ inspects $\{u,v\}$

And it colors v **black!**



USEFUL FACT: PARENTHESIS THEOREM

- **Theorem:** for each pair of nodes u, v
  the intervals of u and v are either **disjoint** or **nested**
- **Proof:** Suppose the intervals are **not disjoint**.
  - Then either $d[v] \in I_u$ or $d[u] \in I_v$
  - WLOG suppose $d[v] \in I_u$
  - Then v is discovered during $DFSVisit(u)$
  - So, v must turn gray after u and black before u
  - So $f[v] < f[u]$
  - So **the intervals are nested**. QED



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edge type	colour of v	discovery/finish times
tree	white	$d[u] < d[v] < f[v] < f[u]$
forward	black	$d[u] < d[v] < f[v] < f[u]$
back	gray	$d[v] < d[u] < f[u] < f[v]$
cross	Q8?	Q7?

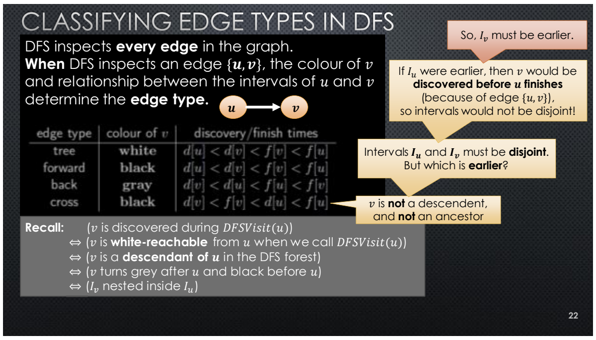
So, I_v must be earlier.

If I_u were earlier, then v would be **discovered before $\boldsymbol{u}$ finishes** (because of edge $\{u,v\}$), so intervals would not be disjoint!

Intervals $\boldsymbol{I_u}$ and $\boldsymbol{I_v}$ must be **disjoint**. But which is **earlier**?

v is **not** a descendent, and **not** an ancestor

Recall: (v is discovered during $DFSVisit(u)$)
⇔ (v is **white-reachable** from u when we call $DFSVisit(u)$)
⇔ (v is a **descendant of $\boldsymbol{u}$** in the DFS forest)
⇔ (v turns grey after u and black before u)
⇔ (I_v nested inside I_u)



CLASSIFYING EDGE TYPES IN DFS

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edge type	colour of v	discovery/finish times
tree	white	$d[u] < d[v] < f[v] < f[u]$
forward	black	$d[u] < d[v] < f[v] < f[u]$
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cross	black	$d[v] < f[v] < d[u] < f[u]$

So, I_v must be earlier.

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Intervals $\boldsymbol{I_u}$ and $\boldsymbol{I_v}$ must be **disjoint**. But which is **earlier**?

v is **not** a descendent, and **not** an ancestor

Recall: (v is discovered during $DFSVisit(u)$)
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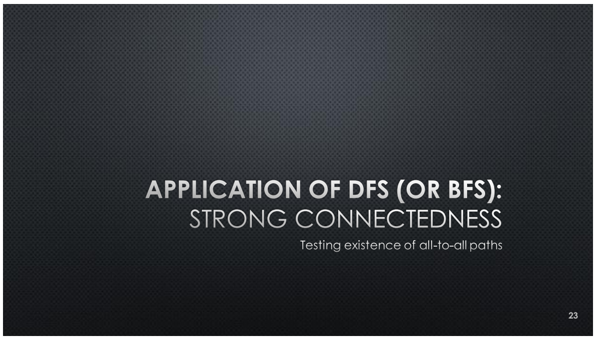
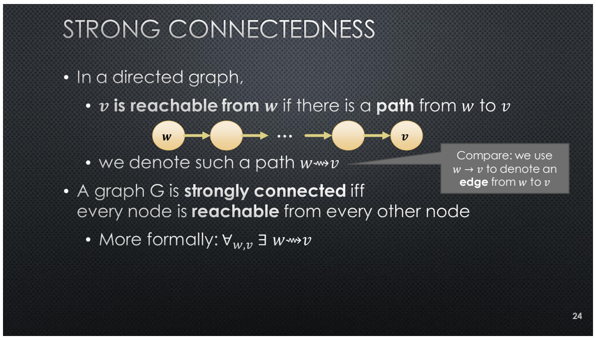


APPLICATION OF DFS (OR BFS): STRONG CONNECTEDNESS

Testing existence of all-to-all paths

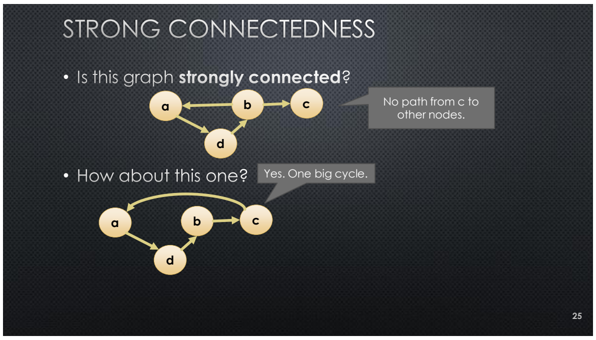


STRONG CONNECTEDNESS

- In a directed graph,
  - **$\boldsymbol{v}$ is reachable from $\boldsymbol{w}$** if there is a **path** from w to v

  - we denote such a path $w \rightsquigarrow v$
- A graph G is **strongly connected** iff every node is **reachable** from every other node
  - More formally: $\forall_{w,v}\ \exists\ w \rightsquigarrow v$

Compare: we use $w \to v$ to denote an **edge** from w to v



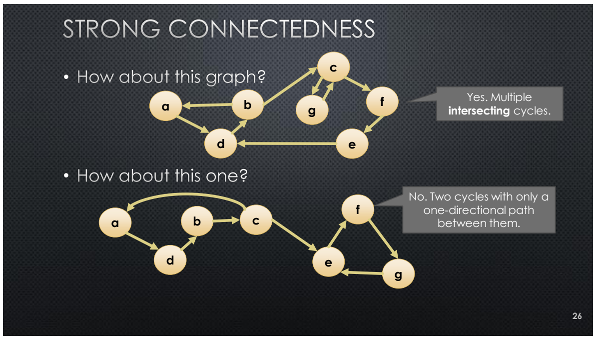
STRONG CONNECTEDNESS

- Is this graph **strongly connected**?

No path from c to other nodes.

- How about this one?

Yes. One big cycle.



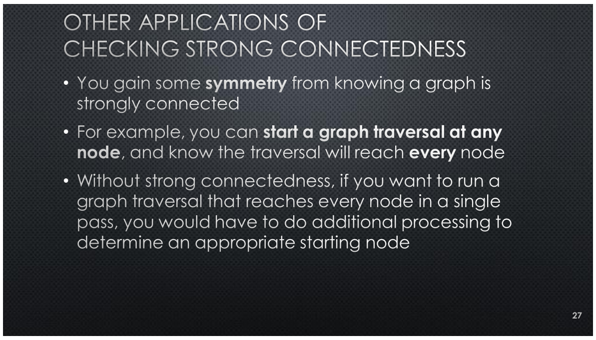
STRONG CONNECTEDNESS

- How about this graph?

Yes. Multiple **intersecting** cycles.

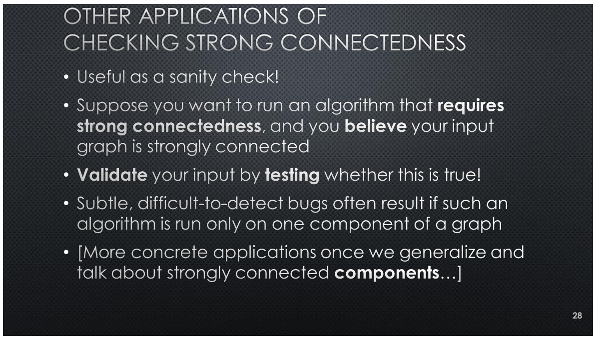
- How about this one?

No. Two cycles with only a one-directional path between them.



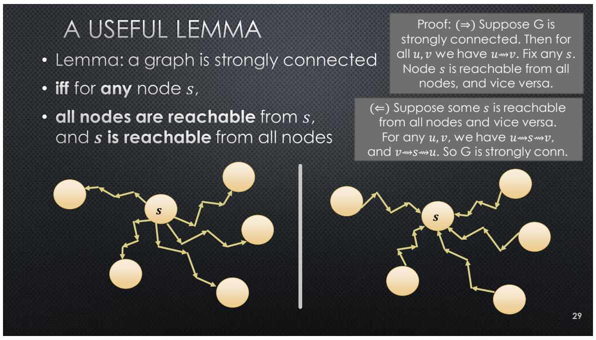
OTHER APPLICATIONS OF CHECKING STRONG CONNECTEDNESS

- You gain some **symmetry** from knowing a graph is strongly connected
- For example, you can **start a graph traversal at any node**, and know the traversal will reach **every** node
- Without strong connectedness, if you want to run a graph traversal that reaches every node in a single pass, you would have to do additional processing to determine an appropriate starting node



OTHER APPLICATIONS OF CHECKING STRONG CONNECTEDNESS

- Useful as a sanity check!
- Suppose you want to run an algorithm that **requires strong connectedness**, and you **believe** your input graph is strongly connected
- **Validate** your input by **testing** whether this is true!
- Subtle, difficult-to-detect bugs often result if such an algorithm is run only on one component of a graph
- [More concrete applications once we generalize and talk about strongly connected **components**…]

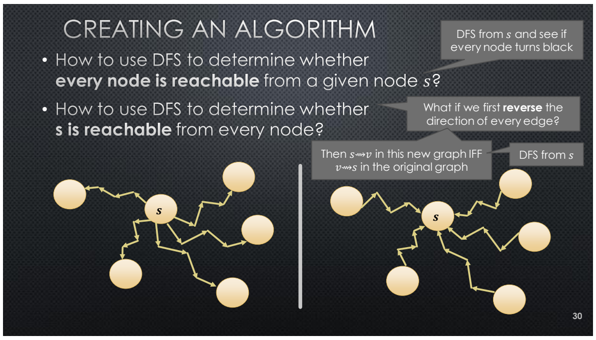


A USEFUL LEMMA

- Lemma: a graph is strongly connected
- **iff** for **any** node s,
- **all nodes are reachable** from s, and s **is reachable** from all nodes

Proof: (⇒) Suppose G is strongly connected. Then for all u, v we have $u \rightsquigarrow v$. Fix any s. Node s is reachable from all nodes, and vice versa.

(⇐) Suppose some s is reachable from all nodes and vice versa. For any u, v, we have $u \rightsquigarrow s \rightsquigarrow v$, and $v \rightsquigarrow s \rightsquigarrow u$. So G is strongly conn.



CREATING AN ALGORITHM

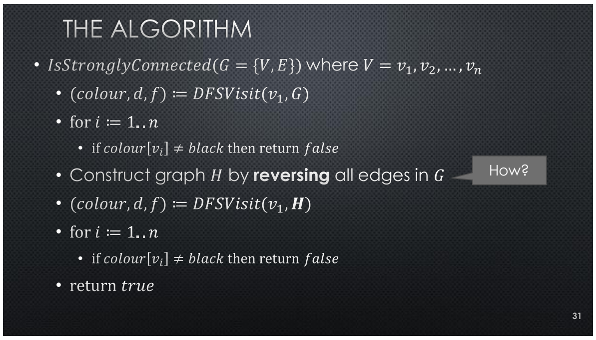
- How to use DFS to determine whether **every node is reachable** from a given node s?
- How to use DFS to determine whether **s is reachable** from every node?

DFS from s and see if every node turns black

What if we first **reverse** the direction of every edge?

Then $s \rightsquigarrow v$ in this new graph IFF $v \rightsquigarrow s$ in the original graph

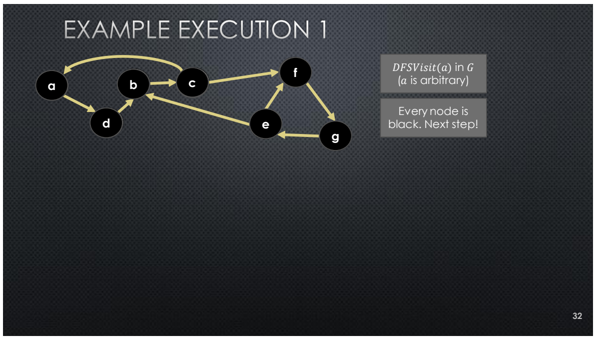
DFS from s



THE ALGORITHM

```
- $IsStronglyConnected(G = \{V, E\})$ where $V = v_1, v_2, \dots, v_n$
    - $(colour, d, f) \coloneqq DFSVisit(v_1, G)$
    - for $i \coloneqq 1..n$
        - if $colour[v_i] \neq black$ then return $false$
    - Construct graph $H$ by reversing all edges in $G$
    - $(colour, d, f) \coloneqq DFSVisit(v_1, \boldsymbol{H})$
    - for $i \coloneqq 1..n$
        - if $colour[v_i] \neq black$ then return $false$
    - return $true$
```

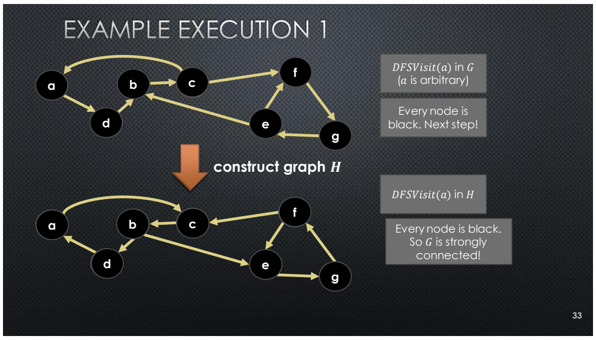
How?



EXAMPLE EXECUTION 1

$DFSVisit(a)$ in G
(a is arbitrary)

Every node is black. Next step!



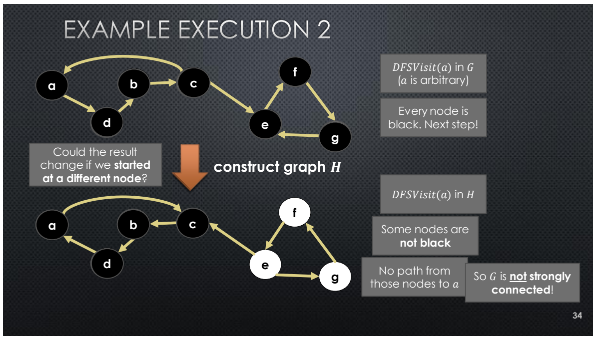
EXAMPLE EXECUTION 1

$DFSVisit(a)$ in G
(a is arbitrary)

Every node is black. Next step!

$DFSVisit(a)$ in H

Every node is black. So G is strongly connected!



EXAMPLE EXECUTION 2

Could the result change if we **started at a different node**?

$DFSVisit(a)$ in G
(a is arbitrary)

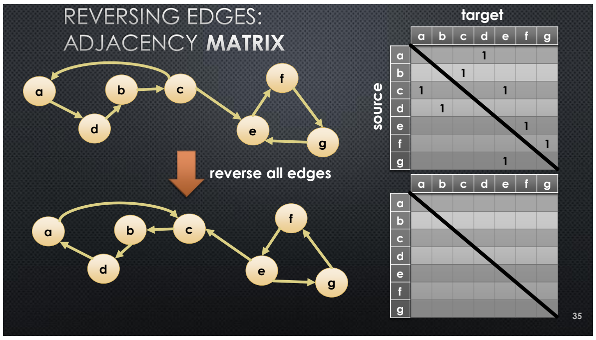
Every node is black. Next step!

$DFSVisit(a)$ in H

Some nodes are **not black**

No path from those nodes to a

So G is **not strongly connected!**

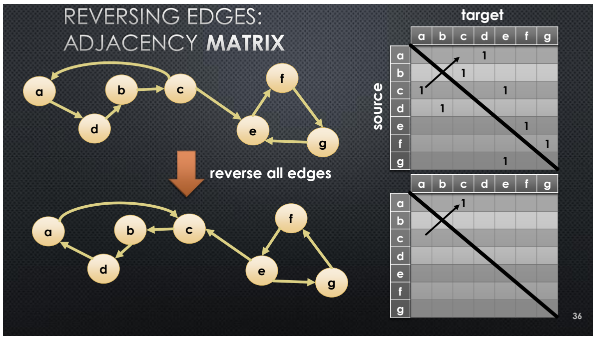


REVERSING EDGES: ADJACENCY **MATRIX**

target

source	a	b	c	d	e	f	g
a				1			
b			1				
c	1				1		
d		1					
e						1	
f							1
g					1		

	a	b	c	d	e	f	g
a							
b							
c							
d							
e							
f							
g							

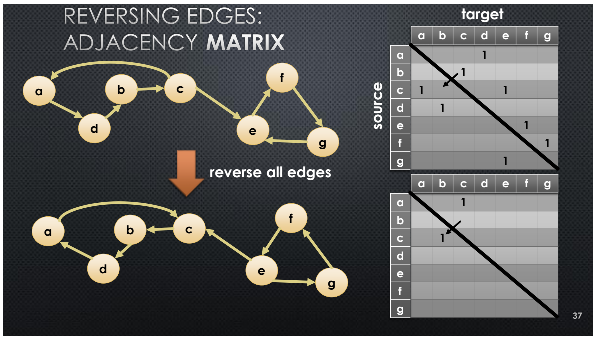


REVERSING EDGES: ADJACENCY **MATRIX**

target

source	a	b	c	d	e	f	g
a				1			
b			1				
c	1				1		
d		1					
e						1	
f							1
g					1		

	a	b	c	d	e	f	g
a			1				
b							
c							
d							
e							
f							
g							



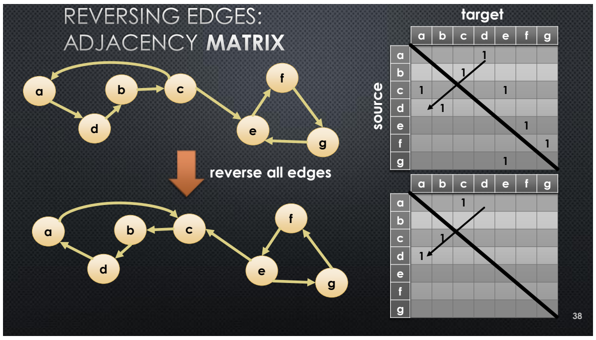
REVERSING EDGES: ADJACENCY **MATRIX**

reverse all edges

target

source	a	b	c	d	e	f	g
a				1			
b			1				
c	1				1		
d		1					
e						1	
f							1
g					1		

	a	b	c	d	e	f	g
a			1				
b							
c		1					
d							
e							
f							
g							



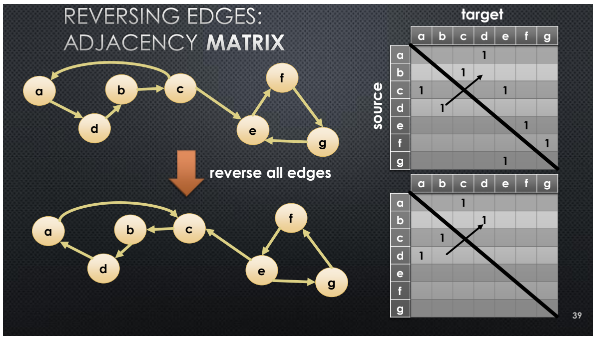
REVERSING EDGES: ADJACENCY **MATRIX**

reverse all edges

target

source	a	b	c	d	e	f	g
a				1			
b			1				
c	1				1		
d		1					
e						1	
f							1
g					1		

	a	b	c	d	e	f	g
a			1				
b							
c		1					
d	1						
e							
f							
g							



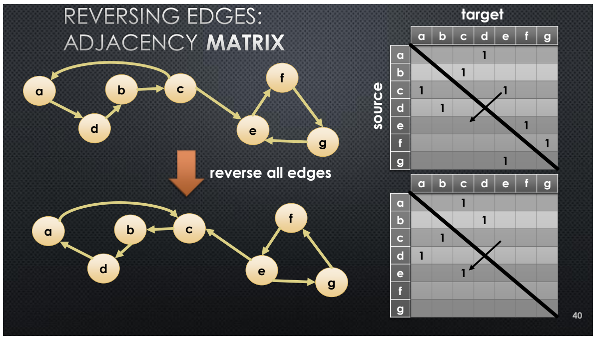
REVERSING EDGES: ADJACENCY **MATRIX**

reverse all edges

target

source	a	b	c	d	e	f	g
a				1			
b			1				
c	1				1		
d		1					
e						1	
f							1
g					1		

	a	b	c	d	e	f	g
a			1				
b				1			
c		1					
d	1						
e							
f							
g							



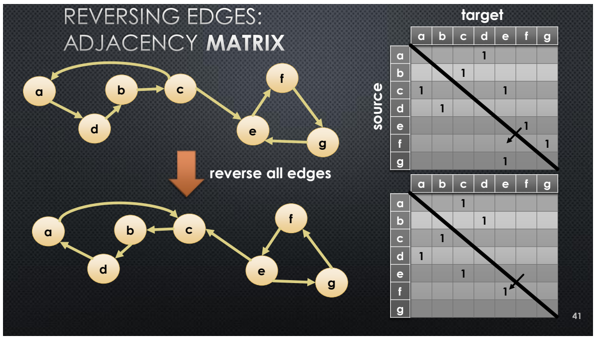
REVERSING EDGES: ADJACENCY **MATRIX**

reverse all edges

target

source	a	b	c	d	e	f	g
a				1			
b			1				
c	1				1		
d		1					
e						1	
f							1
g					1		

	a	b	c	d	e	f	g
a			1				
b				1			
c		1					
d	1						
e			1				
f							
g							



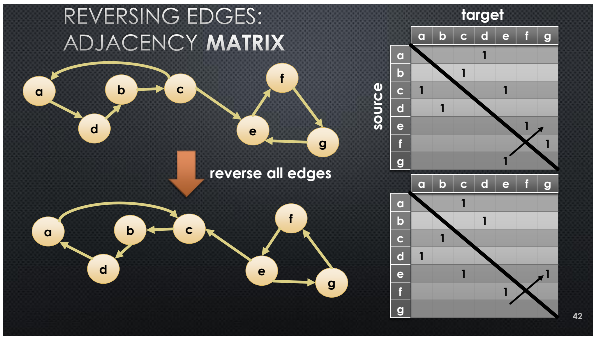
REVERSING EDGES: ADJACENCY **MATRIX**

reverse all edges

target

source	a	b	c	d	e	f	g
a				1			
b			1				
c	1				1		
d		1					
e						1	
f							1
g					1		

	a	b	c	d	e	f	g
a			1				
b				1			
c		1					
d	1						
e			1				
f					1		
g							



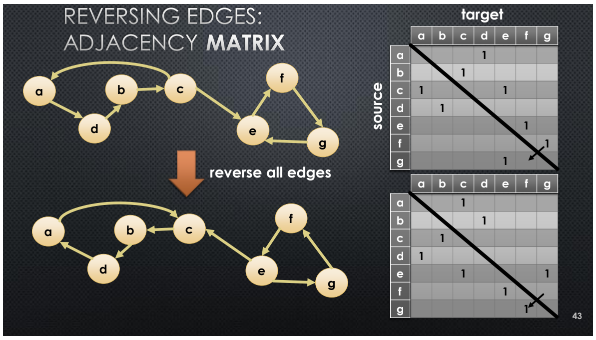
REVERSING EDGES: ADJACENCY **MATRIX**

reverse all edges

target

source	a	b	c	d	e	f	g
a				1			
b			1				
c	1				1		
d		1					
e						1	
f							1
g					1		

	a	b	c	d	e	f	g
a			1				
b				1			
c		1					
d	1						
e			1				1
f					1		
g							

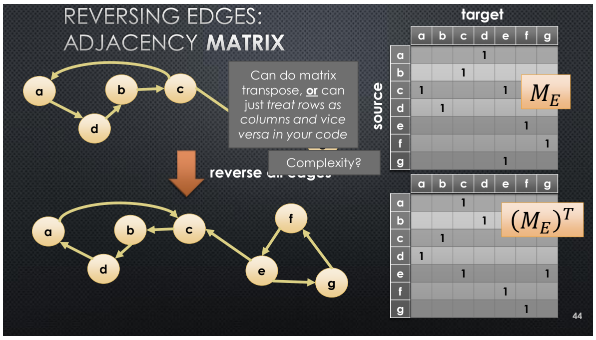


REVERSING EDGES: ADJACENCY **MATRIX**

target

source	a	b	c	d	e	f	g
a				1			
b			1				
c	1				1		
d		1					
e						1	
f							1
g					1		

	a	b	c	d	e	f	g
a			1				
b				1			
c		1					
d	1						
e			1				1
f					1		
g						1	



REVERSING EDGES: ADJACENCY **MATRIX**

Can do matrix transpose, **or** can just *treat rows as columns and vice versa in your code*

Complexity?

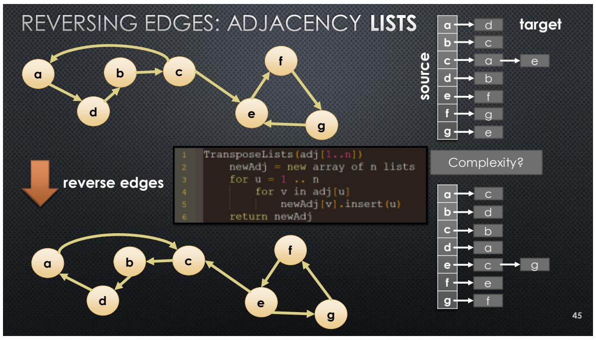
target

source	a	b	c	d	e	f	g
a				1			
b			1				
c	1				1		
d		1					
e						1	
f							1
g					1		

M_E

	a	b	c	d	e	f	g
a			1				
b				1			
c		1					
d	1						
e			1				1
f					1		
g						1	

$(M_E)^T$



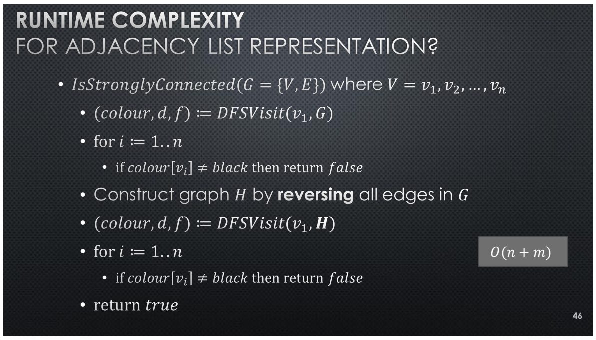
REVERSING EDGES: ADJACENCY **LISTS**

a → d
b → c
c → a → e
d → b
e → f
f → g
g → e

```
TransposeLists(adj[1..n])
    newAdj = new array of n lists
    for u = 1 .. n
        for v in adj[u]
            newAdj[v].insert(u)
    return newAdj
```

Complexity?

a → c
b → d
c → b
d → a
e → c → g
f → e
g → f



RUNTIME COMPLEXITY
FOR ADJACENCY LIST REPRESENTATION?

- $IsStronglyConnected(G = \{V, E\})$ where $V = v_1, v_2, \ldots, v_n$
  - $(colour, d, f) \coloneqq DFSVisit(v_1, G)$
  - for $i \coloneqq 1..n$
    - if $colour[v_i] \neq black$ then return $false$
  - Construct graph H by **reversing** all edges in G
  - $(colour, d, f) \coloneqq DFSVisit(v_1, \boldsymbol{H})$
  - for $i \coloneqq 1..n$
    - if $colour[v_i] \neq black$ then return $false$
  - return $true$

$O(n + m)$