CS 341: ALGORITHMS

Lecture 17: max flow

Readings: CLRS 26.2

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QUICK REVIEW OF LAST TIME

RECALL: MAX-FLOW MIN-CUT THEOREM

- **Theorem 3:** every max s-t flow has value equal to the capacity of a min s-t cut

- We give an **algorithmic proof** of this theorem
  - (showing that one algorithm solves both max-flow and min-cut at the same time)

FORD-FULKERSON METHOD

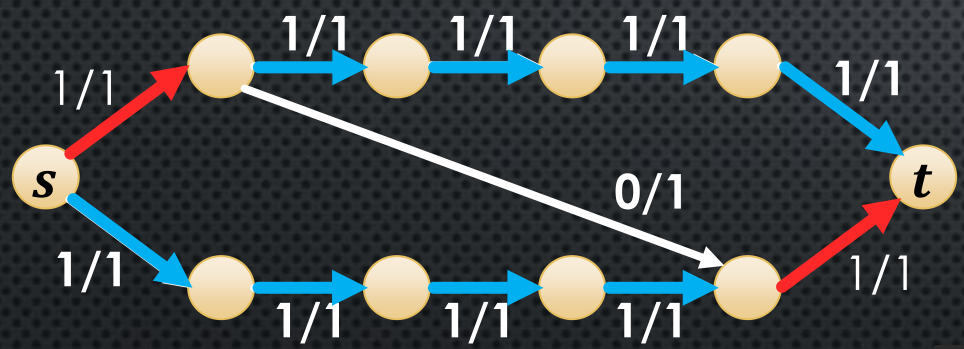
Algorithm development

(mixed together with proof of max-flow min-cut theorem)

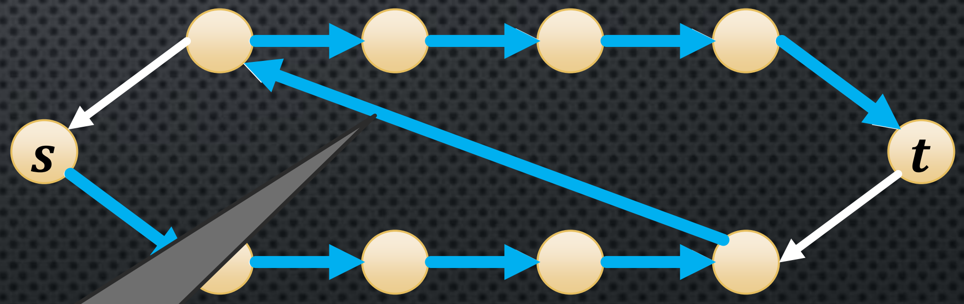
FORD-FULKERSON METHOD

Same Ford as in Bellman-Ford :)

- Can **undo** previous decisions to improve the flow
  - Can effectively "push back" some flow using an **augmenting path** through a **residual graph**



improved flow



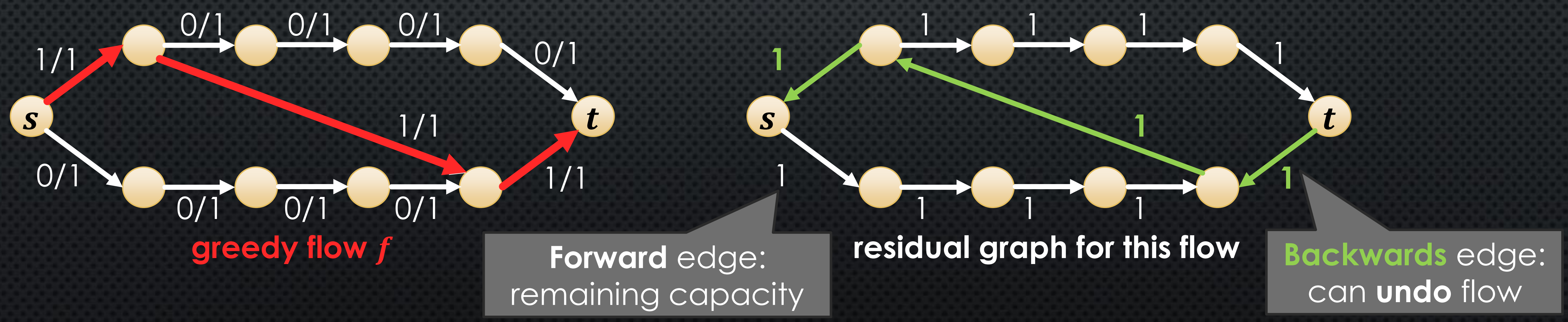
Pushes back the flow on this edge (negating its flow)

"augmenting path"

So, what's the residual graph, how do we find an augmenting path, and how do we improve the flow?

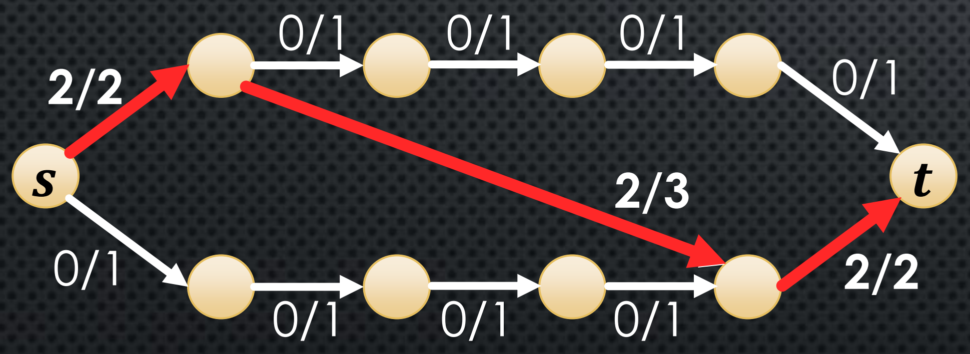
RESIDUAL GRAPH

- A **residual graph** R_f is defined for a **given flow** f and **graph** G
- R_f has the same vertices as G
- For each edge $e = uv$ in G,
  - If $f(e) < c(e)$, then R_f contains a **forward** edge (u, v) with the **remaining capacity** $c(e) - f(e)$
  - If $f(e) > 0$, then R_f contains a **backwards** edge (v, u) with **capacity** $f(e)$ representing flow that could be "pushed back"

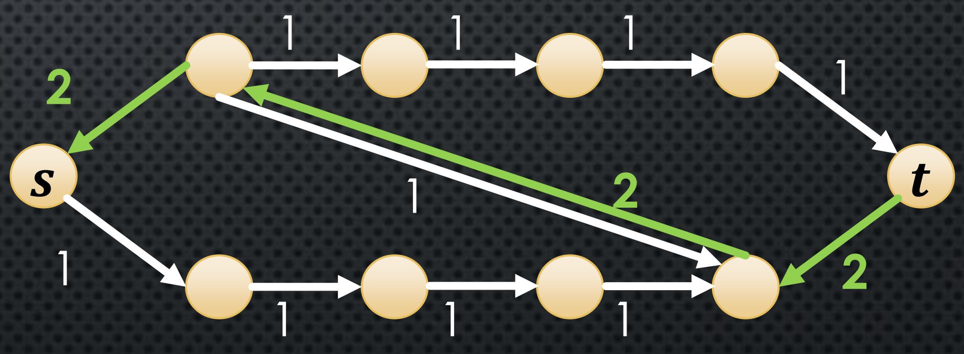


ANOTHER EXAMPLE RESIDUAL GRAPH

- Recall: for each edge $e = uv$ in G,
  - If $f(e) < c(e)$, then R_f contains a **forward** edge (u, v) with the **remaining capacity** $c(e) - f(e)$
  - If $f(e) > 0$, then R_f contains a **backwards** edge (v, u) with **capacity** $f(e)$ representing flow that could be "pushed back"



flow f with value 2

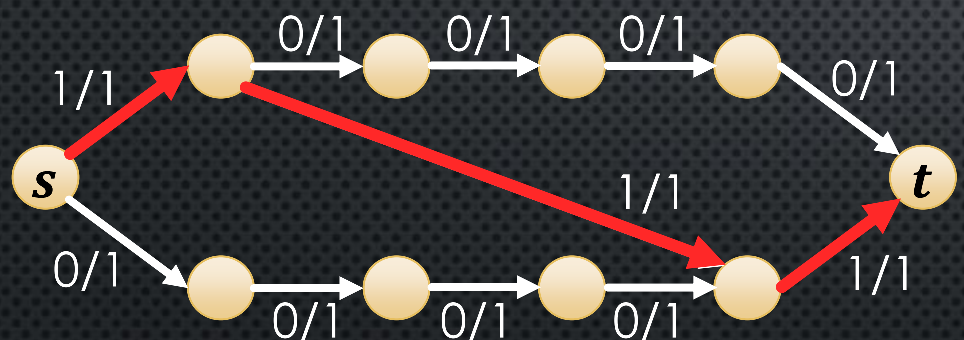


residual graph for this flow

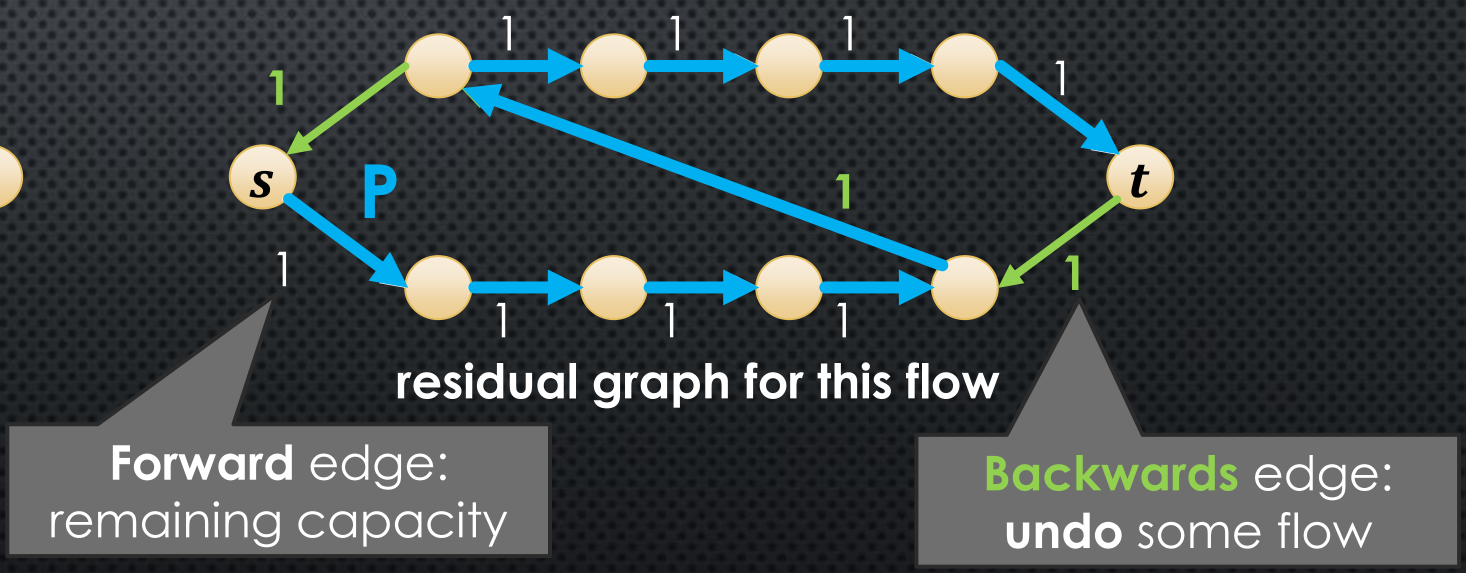
CONTINUING WITH NEW MATERIAL

FORD-FULKERSON METHOD

- Find a **shortest path P** from s to t in the **residual graph**
  - If it **improves** the flow, we call it an **augmenting path**
  - And use it to **update the flow**



greedy flow f

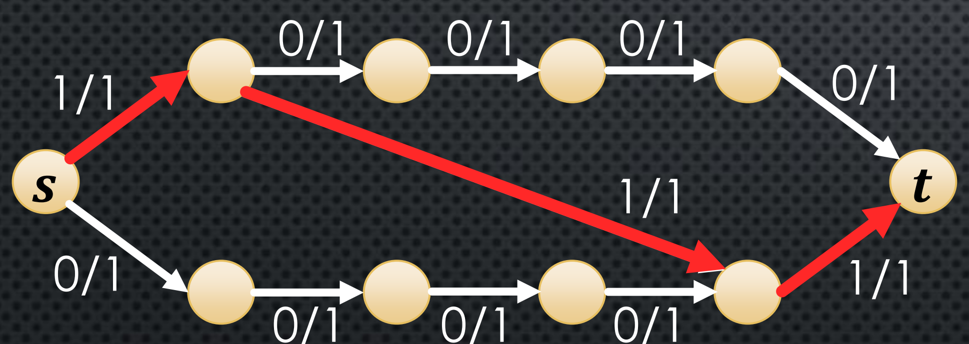


residual graph for this flow

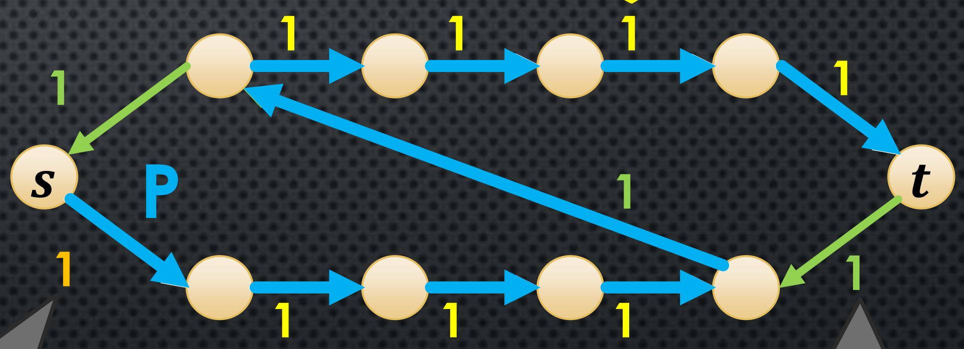
FORD-FULKERSON METHOD

- Find a **shortest path P** from s to t in the **residual graph**
  - If it **improves** the flow, we call it an **augmenting path**
  - And use it to **update the flow**

For each **forward edge** in **P**, **increase** existing flow



greedy flow f



residual graph for this flow

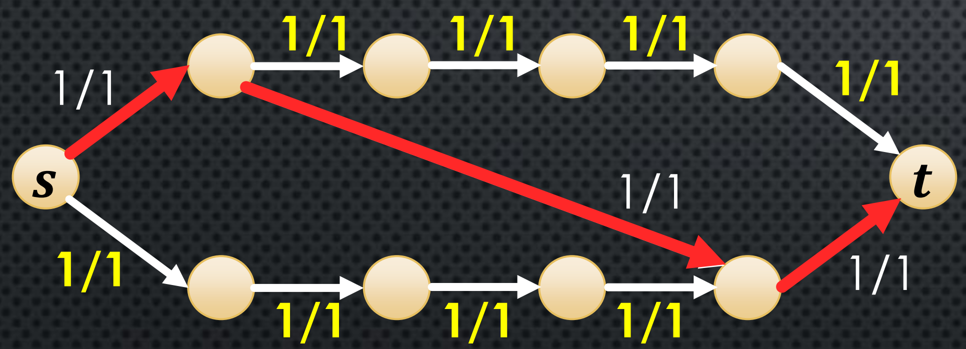
Forward edge: remaining capacity

Backwards edge: **undo** some flow

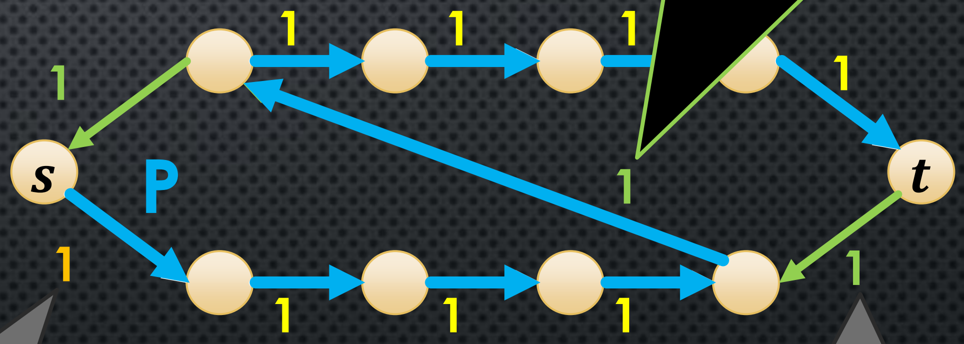
FORD-FULKERSON METHOD

- Find a **shortest path P** from s to t in the **residual graph**
  - If it **improves** the flow, we call it an **augmenting path**
  - And use it to **update the flow**

For each **backwards edge** in **P**, **decrease** existing flow



updated flow



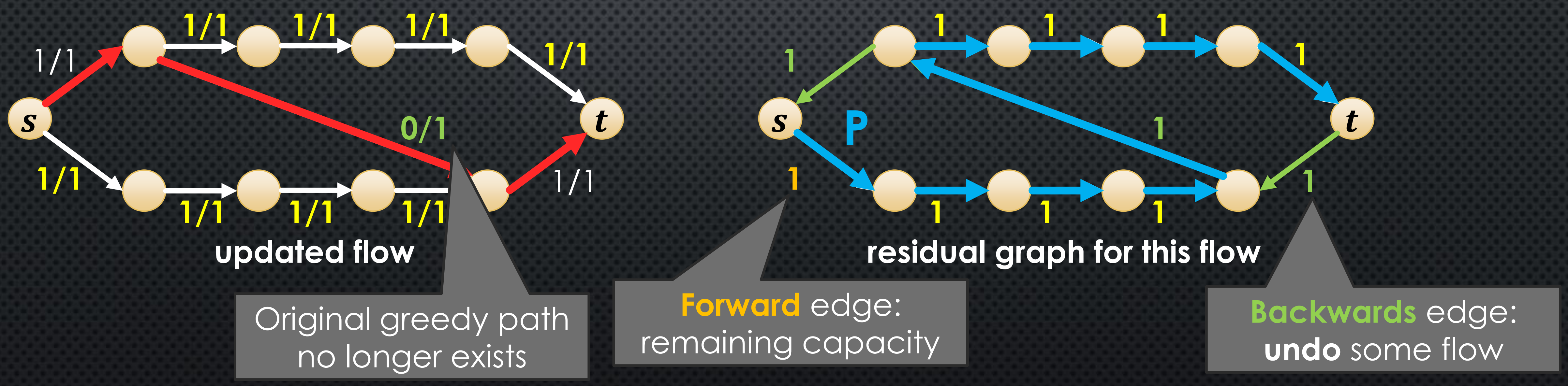
residual graph for this flow

Forward edge:
remaining capacity

Backwards edge:
undo some flow

FORD-FULKERSON METHOD

- Find a **shortest path P** from s to t in the **residual graph**
  - If it **improves** the flow, we call it an **augmenting path**
  - And use it to **update the flow**



FORD-FULKERSON METHOD

- Find a **shortest path P** from s to t in the **residual graph**
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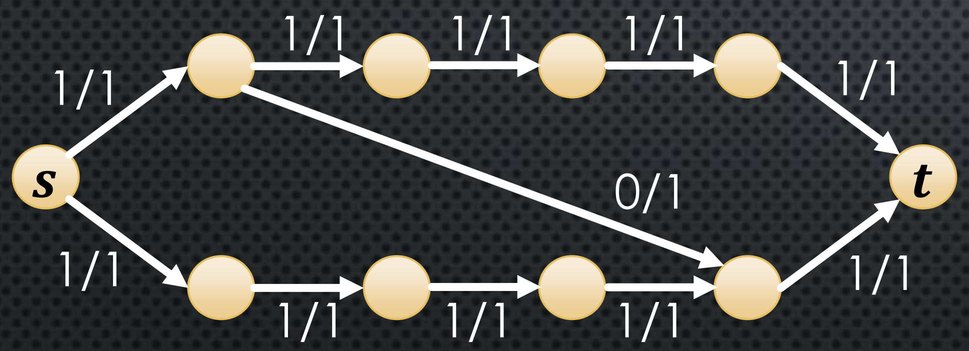
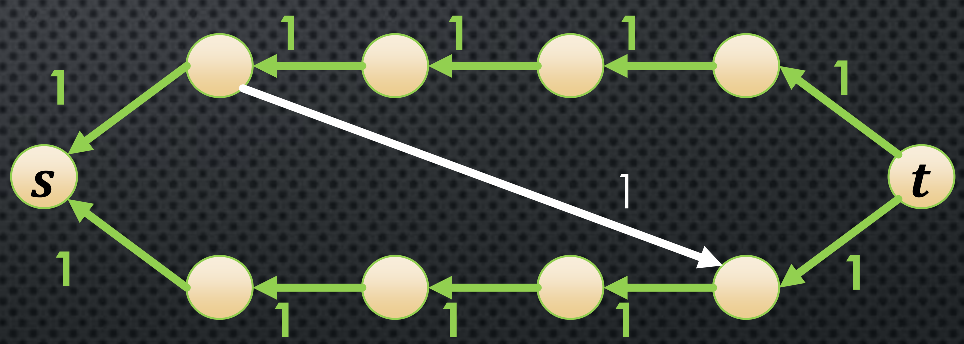


updated flow



new residual graph for this flow

No path from s to t in residual graph. Done!

IMPROVING A FLOW f

GIVEN AN AUGMENTING PATH P

no cycles!

- An augmenting path w.r.t a flow f is a **simple** s-t path in R_f
- Let P be an augmenting path w.r.t f
- Let $\text{bottleneck}(f, P)$ be the minimum capacity of an edge in P
- We show this subroutine $\text{augment}(f, P)$ always improves the value of flow f

```
augment(f, P)
    let b = bottleneck(f, P)
    for each edge e = (u,v) in P
        if e is a forward edge
            f(e) = f(e) + b
        else if e is a backwards edge
            let e' = (v,u)
            f(e') = f(e') - b
```

LEMMA 4: AUGMENT() IMPROVES FLOW f

- Let f be a flow in G with $f^{in}(s) = 0$,
  and P be an augmenting path w.r.t f
- Let f' be the resulting flow after running $\text{augment}(f, P)$
- Then f' **is a flow with** $\textbf{value}(f') = \textbf{value}(f) + \textbf{bottleneck}(f, P)$
- That is, $\textbf{augment}(f, P)$ **increases the flow by** $\textbf{bottleneck}(f, P)$

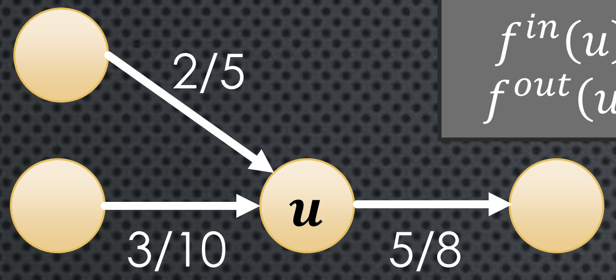
PROOF

- Claim: $\text{augment}(f, P)$ increases the flow by $\text{bottleneck}(f, P)$
- First check f' is a flow
  - Prove capacity and conservation constraints, and $f'^{in}(s) = 0$
- **Are capacity constraints satisfied?**
  - We add/subtract $\text{bottleneck}(f, P)$ to/from each edge
  - And $\text{bottleneck}(f, P)$ is the minimum of the smallest remaining capacity, and the current flow
  - So capacity constraints are satisfied

PROOF

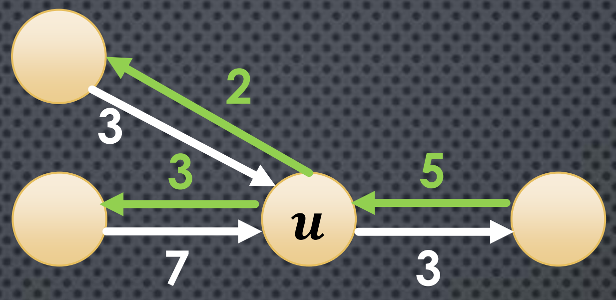
- Claim: $\text{augment}(f, P)$ increases the flow by $\text{bottleneck}(f, P)$
- **How about conservation of flow?**
  - Consider how the flow into and out of each vertex $u \notin \{s, t\}$ changes as a result of running $\text{augment}(f, P)$
  - We show the change in $f^{in}(u)$
    is the same as the change in $f^{out}(u)$
  - There are 4 cases, depending on whether the edges entering/leaving u are **forward** or **backward** edges

Case 1: forward / forward

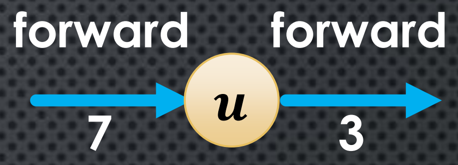


$f^{in}(u) = 5$
$f^{out}(u) = 5$

flow $\boldsymbol{f}$

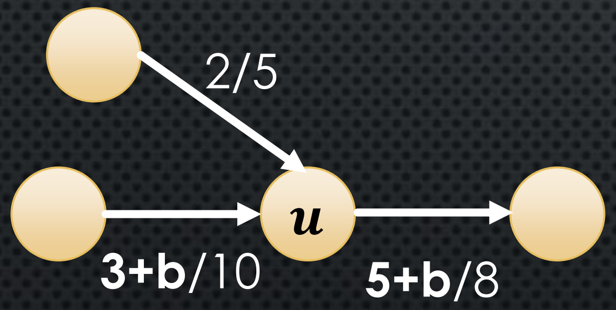


residual graph $\boldsymbol{R_f}$



augmenting path $\boldsymbol{P}$ in $\boldsymbol{R_f}$

Let $\mathbf{bottleneck}(\boldsymbol{f}, \boldsymbol{P}) = \boldsymbol{b}$



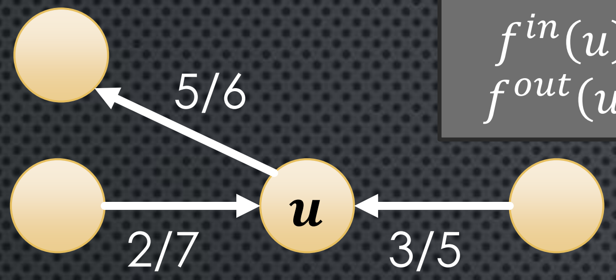
$f'^{in}(u) = 5 + b$
$f'^{out}(u) = 5 + b$

new flow $\boldsymbol{f'}$
(after augmenting)

Both $f^{in}(u)$ and $f^{out}(u)$ are increased by $\text{bottleneck}(f, P)$

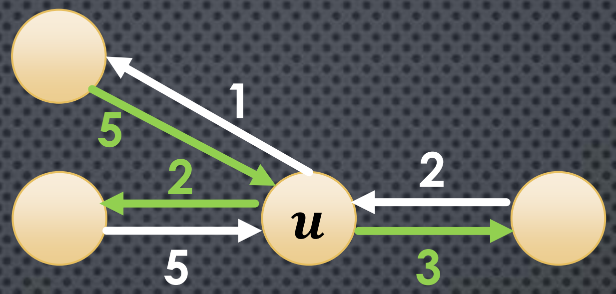
Case 2: backwards / backwards is similar.
Both $f^{in}(u)$ and $f^{out}(u)$ are decreased by $\boldsymbol{b}$

Case 3: forward / backwards

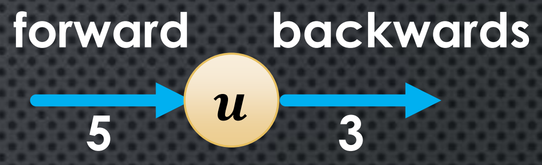


$f^{in}(u) = 5$
$f^{out}(u) = 5$

flow $\boldsymbol{f}$

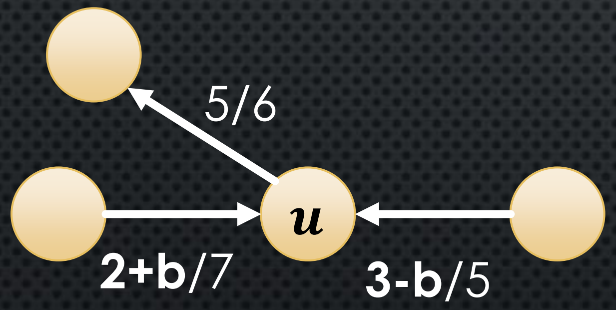


residual graph $\boldsymbol{R_f}$



augmenting path $\boldsymbol{P}$ in $\boldsymbol{R_f}$

Let $\mathbf{bottleneck}(\boldsymbol{f}, \boldsymbol{P}) = \boldsymbol{b}$



$f'^{in}(u) = 5$
$f'^{out}(u) = 5$

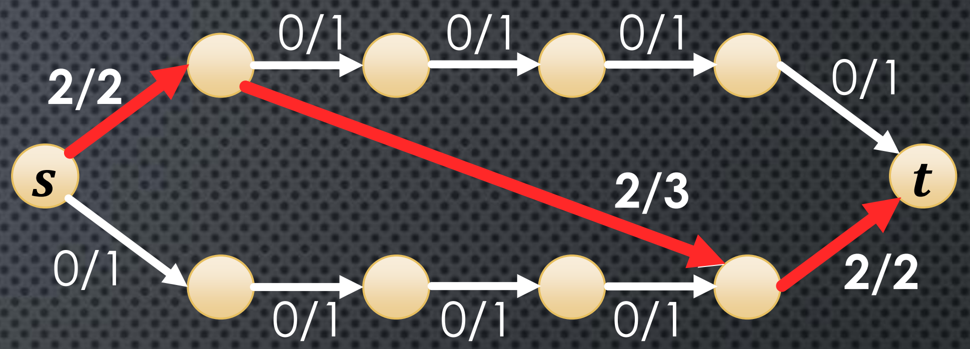
new flow $\boldsymbol{f'}$
(after augmenting)

Added and subtracted
$\boldsymbol{b}$ terms **cancel out**

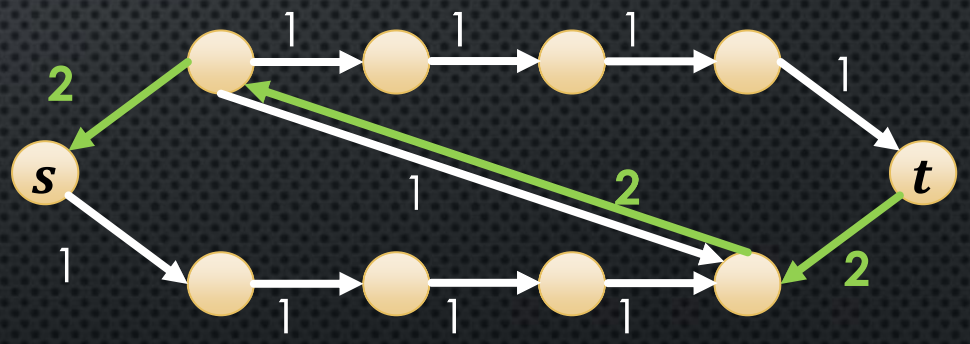
Case 4: backwards / forwards is similar.

SHOWING ${f'}^{in}(s) = 0$

- Last step in showing f' is a flow
  - Prove: s still has no flow into it
- Since f is a flow, $f^{in}(s) = 0$
- To get ${f'}^{in}(s) > 0$, an augmenting path must include an edge **into s**
- But then an augmenting path starts at s, then returns to s, forming a cycle -- contradiction!



flow f with value 2



residual graph for this flow

FINISHING LEMMA 4: AUGMENT() IMPROVES FLOW

- Finally we argue $\text{value}(f') = \text{value}(f) + \text{bottleneck}(f, P)$
- f and f' are flows, so $\text{value}(f') = f'^{out}(s)$ and $value(f) = f^{out}(s)$
- We thus show $f'^{out}(s) = f^{out}(s) + \text{bottleneck}(f, P)$
- The augmenting path P is a **simple** path (leaving s exactly once)
- And there is no flow into s,
  so the edge $e \in P$ leaving s is a **forward edge**
- This means $\text{augment}(f, P)$ **adds** $\text{bottleneck}(f, P)$ to $f(e)$
- So $f'^{out}(s) = f^{out}(s) + \text{bottleneck}(f, P)$

FORD-FULKERSON METHOD

- By Lemma 4, starting from any flow f,
  if we can **find an augmenting path** P w.r.t f in R_f,
  then we can use $\text{augment}(f, P)$ to **improve our flow**
- Ford-Fulkerson does this repeatedly **starting** from an **empty flow**

```
FordFulkerson(G=(V,E))
    for e in E
        f(e) = 0

    while there is a simple s-t path P in Rf do
        augment(f, P)
        and update the residual graph Rf
```

What we have proved so far: **augmenting improves flow.**

We **don't know yet** if

1. **we can actually obtain the max flow**, or
2. **whether max-flow = min-cut.**

MAX-FLOW MIN-CUT THEOREM PROOF

PROOF STRATEGY

- Claim: when there is **no augmenting path**, there is a **cut with capacity equal to** the value of **the current flow**.
- Proving this will simultaneously
  - prove the max-flow min-cut theorem,
  - prove correctness of the Ford-Fulkerson method,
  - solve the max flow problem, and
  - solve the min cut problem

PROVING MAX FLOW = MIN CUT

Two directions:
max flow $\leq$ **min cut** and **max flow** $\geq$ **min cut**

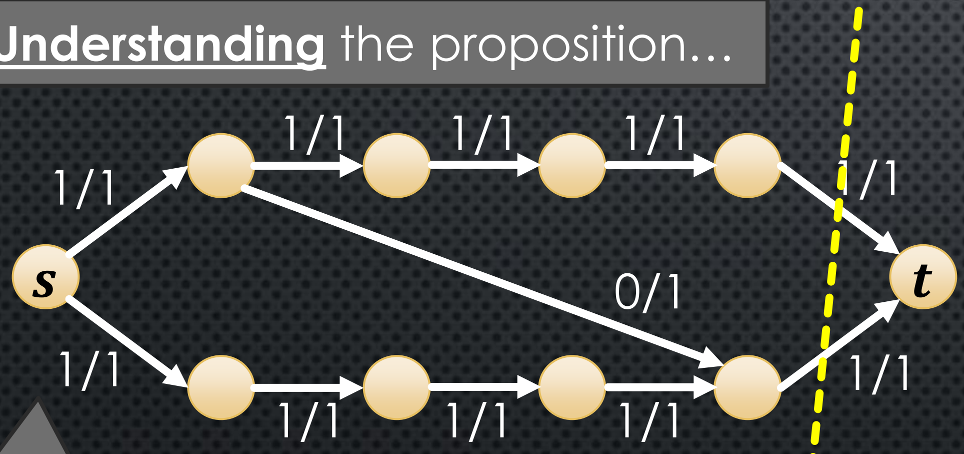
We actually proved the $\leq$ direction already (**Lemma 2 last time**) when discussing upper bounds for max flow

It remains to prove the $\geq$ direction.

PROVING MAX FLOW ≥ MIN CUT

- Proposition: if f is an s-t flow such that
  there is no s-t path in the residual graph R_f,
  then there is an s-t cut S s.t. $\text{value}(f) = c^{out}(S)$

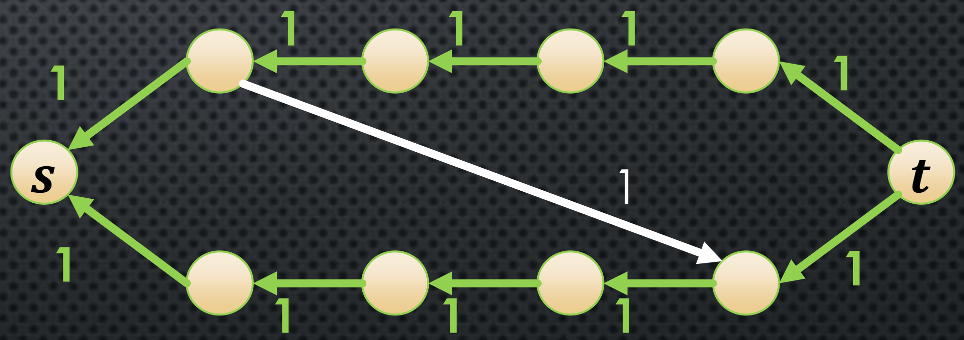
<u>Understanding</u> the proposition…



flow f

If flow value = 2

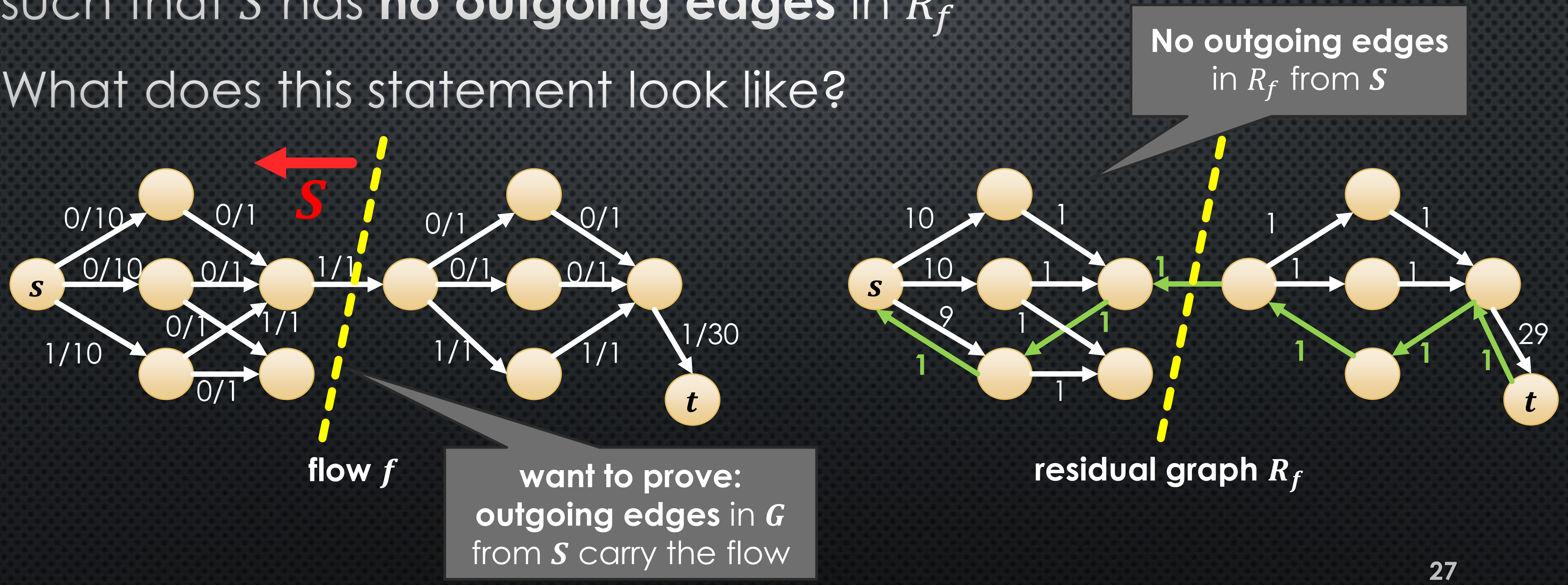
then cut exists with capacity 2 = flow value



residual graph R_f
containing **no** s-t paths

PROVING THE PROPOSITION

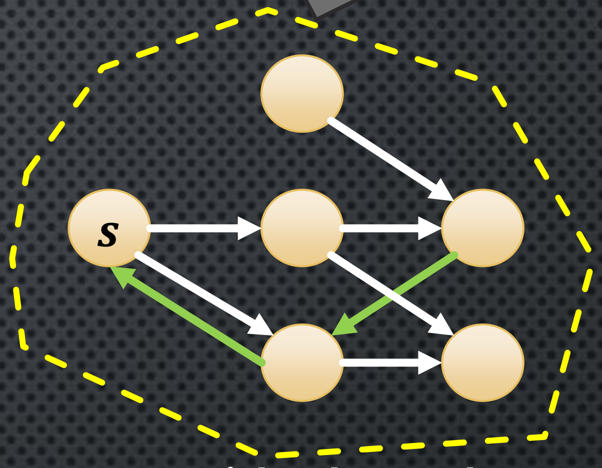
- Since there is no s-t path in R_f, there is a subset S of vertices with $s \in S$, $t \notin S$ such that S has **no outgoing edges** in R_f
- What does this statement look like?



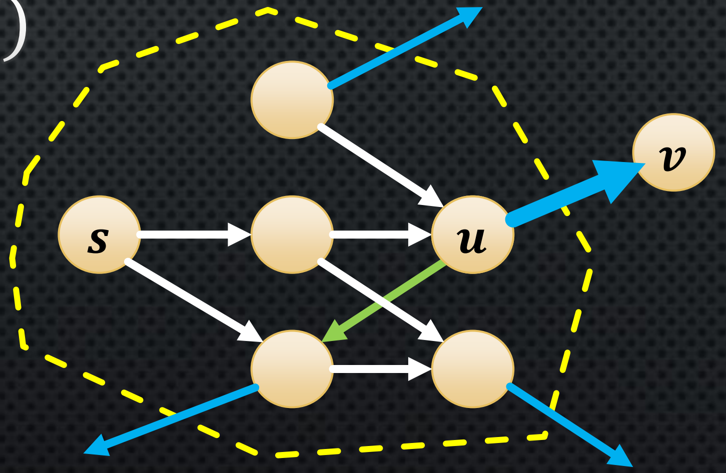
PROVING THE PROPOSITION

- Since there is no s-t path in R_f,
  there is a subset S of vertices with $s \in S$, $t \notin S$
  such that S has **no outgoing edges** in R_f
- Claim: $c^{out}(S) = \text{value}(f)$
- Consider two types of edges. Type 1:
  - uv **exiting S in G** ($uv \in \delta^{out}(S)$ in G, $u \in S, v \notin S$)
  - Since S has no outgoing edge in R_f,
    we know $uv \notin R_f$
  - This implies $f(uv) = c(uv)$, as otherwise
    uv would be a forward edge in R_f

No outgoing edges in R_f from S



residual graph R_f

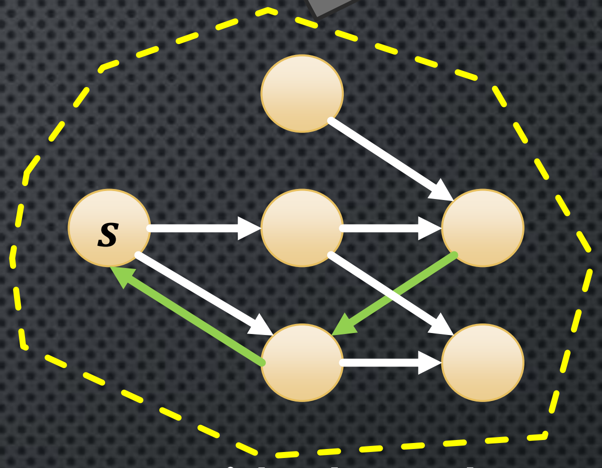


flow f in G

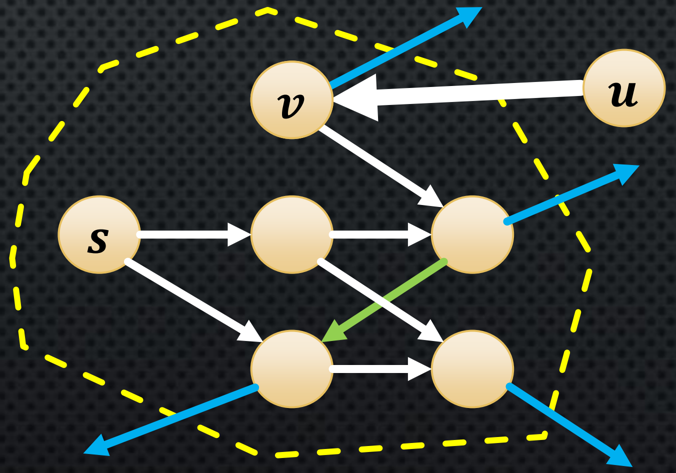
PROVING THE PROPOSITION

- Claim: $c^{out}(S) = \text{value}(f)$
- Consider two types of edges. Type 2:
  - uv **entering** S **in** G ($uv \in \delta^{in}(S)$ in G, $u \notin S, v \in S$)
  - Since S has no outgoing edge in R_f, we know there is no edge $vu \notin R_f$ (note vu would be directed out of S)
  - This implies $f(uv) = 0$, as otherwise vu would be a backwards edge in R_f

No outgoing edges in R_f from S



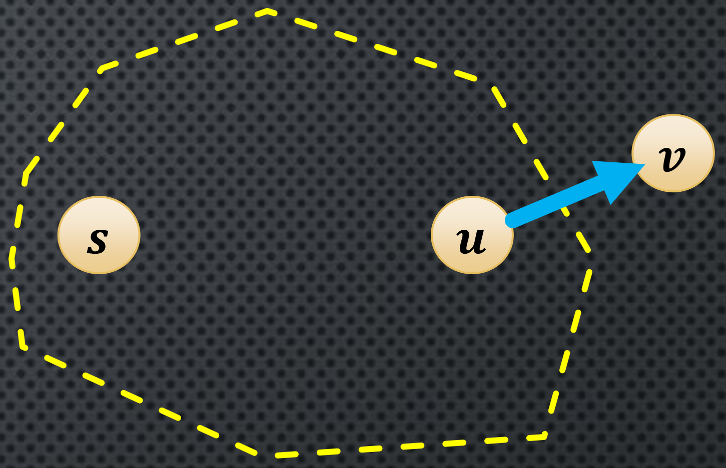
residual graph R_f

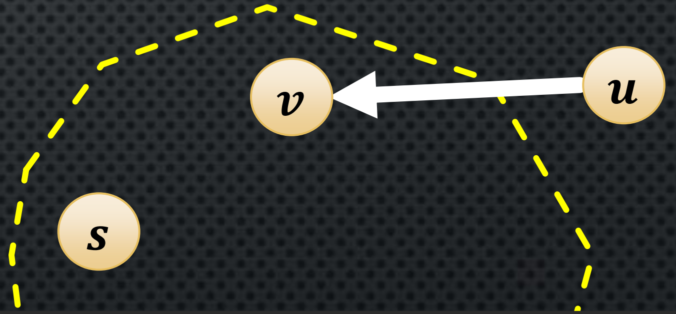


flow f in G

PROVING THE PROPOSITION

- We just showed
  - For edge uv directed out of S, $f(uv) = c(uv)$
  - For edge uv directed into S, $f(uv) = 0$
- So $f^{out}(S) - f^{in}(S) = c^{out}(S) - 0 = c^{out}(S)$
- This proves the proposition. I.e., given flow f, if there are no s-t paths in R_f, then **there is a cut matching the flow**





Note this was the last thing remaining to prove the min-cut max-flow theorem, and the correctness of Ford-Fulkerson

TIME COMPLEXITY

of the Ford-Fulkerson method

RUNTIME OF FORD-FULKERSON

- Depends on the implementation

```
FordFulkerson(G=(V,E))
    for e in E
        f(e) = 0

    while there is a simple s-t path P in Rf do
        augment(f, P)
        and update the residual graph Rf
```

- How do we find an augmenting path?
- How many times do we need to augment before we terminate?

RUNTIME OF FORD-FULKERSON

- Assume we find any arbitrary augmenting path P, using any technique, in $O(n + m)$ time
- Then every time $\text{augment}(f, P)$ is run, we know only that the flow **increases**

If capacities are reals (and in particular some are irrational), this may **never** terminate!

- If capacities are **integers**, the increase is at least 1
- In this case, **if max flow is k then runtime is $O(k(n + m))$**
  - For max flow we assume a connected graph, so this is $O(km)$
  - **Very bad if k is large**

WORST CASE FOR THIS APPROACH

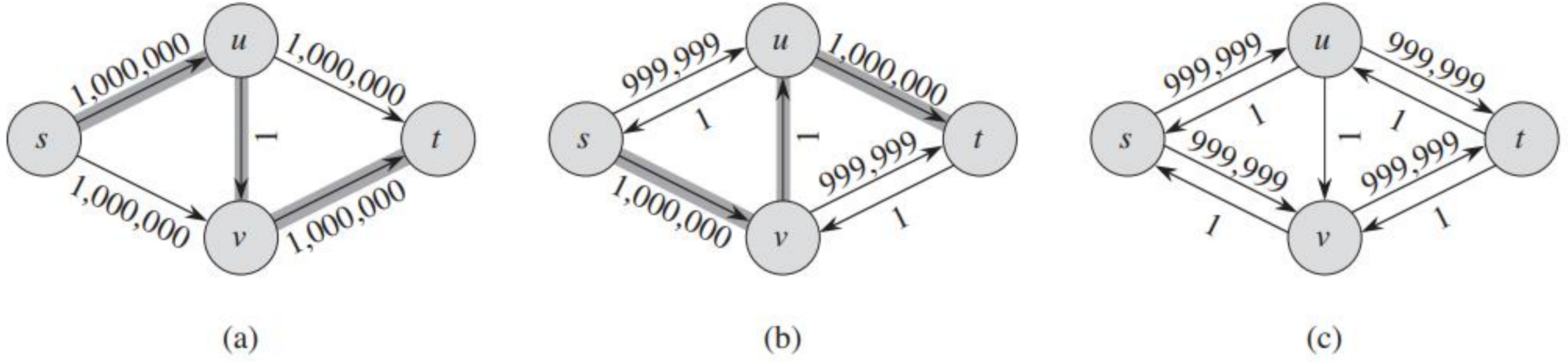


Figure 26.7 **(a)** A flow network for which FORD-FULKERSON can take $\Theta(E\,|f^*|)$ time, where f^* is a maximum flow, shown here with $|f^*| = 2{,}000{,}000$. The shaded path is an augmenting path with residual capacity 1. **(b)** The resulting residual network, with another augmenting path whose residual capacity is 1. **(c)** The resulting residual network.

EDMONDS-KARP APPROACH

- **Use BFS** to find a shortest path (in terms of number of edges) and use that as an augmenting path
- It turns out this always **terminates after $\boldsymbol{O(nm)}$ augmenting paths**
  - (even with real capacities)
- BFS takes $O(n+m)$ time; $\boldsymbol{O(m)}$ since the graph is connected
- **So total runtime is $\boldsymbol{O(nm^2)}$**

There are more sophisticated algorithms with $O(V^2E)$ and even $O(V^3)$ runtimes (**optional**: CLRS 26.4, 26.5)

In 2022, researchers found *an almost* linear time algorithm, which leverages techniques from convex optimization and sophisticated data structures