# CS 341: ALGORITHMS

**Lecture 22: Intractability V – More NPC transformations**

Readings: see website

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## LAST TIME

- Polynomial transformations
  - Poly transformation from Clique to Vertex Cover
- NP Completeness
  - SAT is NP complete (NPC)
  - Got part way through showing 3SAT is NPC
    - Did poly transformation from SAT to 3SAT
    - Need to also show 3SAT is in NP

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## LET'S DO A BRIEF REVIEW

of **NPC**, **poly transformations**, and showing a **problem is in NP**

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## COMPLEXITY CLASS **NP-COMPLETE** (NPC)

The complexity class **NPC** denotes the set of all decision problems $\Pi$ that satisfy the following two properties:

$\Pi \in$ **NP**

For all $\Pi' \in$ **NP**, $\Pi' \leq_P \Pi$.

**NPC** is an abbreviation for **NP-complete**.

Note that the definition does not imply that NP-complete problems exist!

**Mechanics of proving $\Pi \in$ NPC**
1. Show $\Pi$ is in NP
2. Show a poly transformation from some NPC problem to $\Pi$

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## MECHANICS OF SHOWING A PROBLEM IS **IN NP**

- How to show $\Pi \in NP$

1. Define a yes-certificate
2. Design a poly-time $verify(I, C)$ algorithm
3. Correctness proof
   - **Case 1:** Let $I$ be any yes-instance; Find $C$ such that $verify(I, C) = true$
   - **Case 2:** Let $I$ be any no-instance, and $C$ be any certificate; Prove $verify(I, C) = false$

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## POLYNOMIAL TRANSFORMATION FOR PROVING $\Pi_2$ IS IN NPC

Known NPC problem

Problem you want show is NPC

- Let $\Pi_1$ and $\Pi_2$ be decision problems
- $\Pi_1 \leq_P \Pi_2$ **iff** there exists $f : \mathcal{I}(\Pi_1) \to \mathcal{I}(\Pi_2)$ such that:
  - $f(I)$ is computable in poly-time, for all $I \in \mathcal{I}(\Pi_1)$
  - If $I \in \mathcal{I}_{yes}(\Pi_1)$ then $f(I) \in \mathcal{I}_{yes}(\Pi_2)$
  - **If $f(I) \in \mathcal{I}_{yes}(\Pi_2)$ then $I \in \mathcal{I}_{yes}(\Pi_1)$**

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# LET'S FINISH SHOWING 3SAT ∈ NPC

- Already poly transformed SAT to 3SAT
- Need to show 3SAT in NP



# PROVING 3SAT IS **IN NP**

1. Define desired YES-certificate
2. Design a poly-time $verify(I, C)$ algorithm
3. Correctness proof
   - **Case 1:** Let $I$ be any yes-instance; Find $C$ such that $verify(I, C) = true$
   - **Case 2:** Let $I$ be any no-instance, and $C$ be any certificate; Prove $verify(I, C) = false$
   - **Contrapositive of case 2:** Suppose $verify(I, C) = true$; Prove $I$ is a yes-instance

**3SAT input $I = (Clauses[1..m], n)$:** a list of $m$ **clauses**, and the number $n$ of variables. Each clause contains literals. Each literal is a pair (var, neg): a variable $\in \{1..n\}$ & a negation bit

YES-certificate $C$ = array with one bit per variable in $\{1..n\}$ representing a **satisfying assignment**

```
verify3SAT(I=(Clauses[1..m], n), C)
    if C is not an array of n bits return false

    numSat = 0
    for each c in Clauses
        for each literal (var, neg) in c
            if (C[var] && !neg) or (!C[var] && neg)
                numSat++
                break

    return (numSat == m)
```

This takes $O(|\text{Clauses}|)$ time, which is polynomial in $\text{Size}(I)$



# MECHANICS OF SHOWING A PROBLEM IS **IN NP**

1. Define desired YES-certificate
2. Design a poly-time $verify(I, C)$ algorithm
3. Correctness proof
   - **Case 1:** Let $I$ be any yes-instance; Find $C$ such that $verify(I, C) = true$
   - **Case 2:** Let $I$ be any no-instance, and $C$ be any certificate; Prove $verify(I, C) = false$
   - **Contrapositive of case 2:** Suppose $verify(I, C) = true$; Prove $I$ is a yes-instance

Let $I$ be a yes-instance of 3SAT. Then it has a satisfying assignment $\boldsymbol{A_s}$. And, $verify(I, A_s)$ will see that each clause contains a literal satisfied by this assignment, so $verify$ will see $numSat = |Clauses|$ and return true.

Suppose $verify(I, C)$ returns true. Then $numSat = |Clauses|$, so $numSat$ was incremented in each iteration of the loop over clauses, so each clause contains a satisfied literal, so the 3SAT formula in $I$ is satisfied by $C$, so $I$ is a yes-instance.

It follows that **3SAT is in NP.** Since we have already shown SAT $\leq_P$ 3SAT, we now know that **3SAT is NP-COMPLETE.**



# Summary of Polynomial Transformations

SAT ← Every problem in NP can be poly transformed to

↓ can be poly transformed to

3-SAT

Since SAT is NP-complete, so is 3-SAT!

Today and next time let's start filling out a hierarchy of reductions that prove several problems are NP complete

But first, since you need to know **NP hardness** for your assignment…



# NP-HARDNESS

*Intuitively: problems that are **at least as hard** as NP-complete (but are not necessarily decision problems)*



# NP-hard Problems

**TSP-Optimal Value** is also NP-hard (and not in NP) — This version returns the **total weight** of an optimal Hamiltonian cycle

A problem $\Pi$ is **NP-hard** if there exists a problem $\Pi' \in$ **NPC** such that $\Pi' \leq_P^T \Pi$.

Every NP-complete problem is automatically NP-hard, but there exist NP-hard problems that are not NP-complete.

Typical examples of NP-hard problems are optimization problems corresponding to NP-complete decision problems.

For example, **TSP-Decision** $\leq_P^T$ **TSP-Optimization** and **TSP-Decision** $\in$ **NPC**, so **TSP-Optimization** is NP-hard.

Reduction from lecture 19/20

Returns an **optimal Hamiltonian cycle**

## COMPARING NPC AND NP HARD

- $\Pi \in$ NPC
  - Must be a decision problem
  - Must poly transform some NPC problem to $\Pi$
  - Must show $\Pi$ in NP
- $\Pi \in$ NPHard
  - Does not need to be a decision problem
  - Can use either poly transform or poly Turing reduction
  - Does not need to be in NP (and can't be if not decision)

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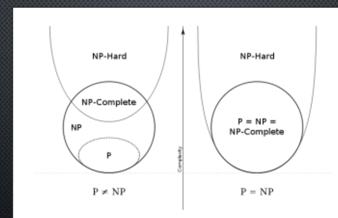
## TWO POSSIBLE REALITIES…

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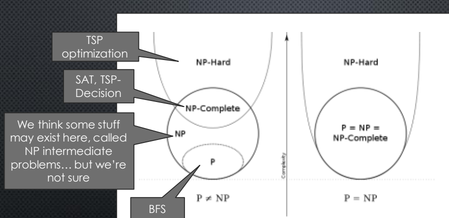
## SOME PROBLEMS IN EACH

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## ESTABLISHING ANOTHER NPC PROBLEM
## … BY TRANSFORMING **3-SAT** TO **CLIQUE**

(Proving 3-SAT $\leq_P$ Clique)

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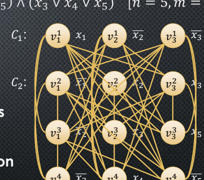
## SHOWING 3-SAT $\leq_P$ CLIQUE

- Let $I$ be an instance of 3-SAT with $n$ variables $x_1 \ldots x_n$ and $m$ clauses $C_1 \ldots C_m$
  - *E.g.,* $(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3 \vee x_5) \wedge (\overline{x_3} \vee x_4 \vee \overline{x_5})$ $[n = 5, m = 4]$
- We construct **Clique** input $\boldsymbol{f(I) = (G, k)}$ :
  - Node $v_\ell^c$ for each literal $1 \leq \ell \leq 3$ in each clause $1 \leq c \leq m$ (so $|V| = 3m$)
  - Edges between all **non-contradictory** pairs of nodes (no $x_i \wedge \overline{x_i}$) in **different clauses**
  - $k = m$ (can we find an $\boldsymbol{m}$-clique?)
  - Must prove this is a **polynomial transformation**




**Reasonable 3-SAT representation:** $array[1..m]$ of clauses <$l_1, l_2, l_3$> of literals <$v, neg$> where $v \in \{1..n\}$.

Note $O(m) \subseteq O(Size(I))$,
So runtime $O(m^2) \subseteq O(Size(I)^2)$ ➔ **polytime!**

**Runtime:** create $3m$ nodes, $O(m^2)$ edges, at $O(1)$ time each

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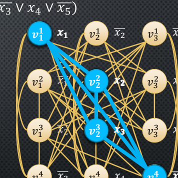
## SHOWING 3-SAT $\leq_P$ CLIQUE

- Let $I$ be an instance of 3-SAT with $n$ variables $x_1 \ldots x_n$ and $m$ clauses $C_1 \ldots C_m$
  - *E.g.,* $(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3 \vee x_5) \wedge (\overline{x_3} \vee x_4 \vee \overline{x_5})$
- **Case 1:** Suppose $I$ is a **yes**-instance of 3-SAT, and show $\boldsymbol{f(I)}$ is a **yes**-instance of $\boldsymbol{m}$**-clique**
- Since $I$ is a yes-instance, $\exists$ a **satisfying** assignment
  - *E.g.,* $x_1 = 1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 0,\ x_5 = 0$
- For each clause $\boldsymbol{C_i}$, let $\boldsymbol{s_i}$ be a **satisfied literal** in $\boldsymbol{C_i}$
  - *E.g.,* $\boldsymbol{s_1} = x_1,\ \boldsymbol{s_2} = x_2,\ \boldsymbol{s_3} = x_3,\ \boldsymbol{s_4} = \overline{x_5}$




- **Claim:** the *corresponding nodes* form an $\boldsymbol{m}$**-clique**
  - There are $m$ of these nodes, each in a different clause
  - None of them represent contradictory truth assignments
  - So, there are edges between all pairs of them → they form an $m$-clique

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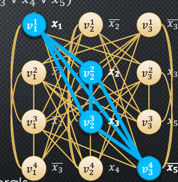
## SHOWING 3-SAT $\leq_P$ CLIQUE

- Let $I$ be an instance of 3-SAT with $n$ variables $x_1 \ldots x_n$ and $m$ clauses $C_1 \ldots C_m$
  - *E.g.,* $(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3 \vee x_5) \wedge (\overline{x_3} \vee x_4 \vee \overline{x_5})$
- **Case 2:** Suppose $\boldsymbol{f(I)}$ is a **yes**-instance of $m$-clique, and show $I$ is a **yes**-instance of **3-SAT**
- Since $f(I)$ is a yes-instance, it contains an $m$-clique
- Clique contains edges between all pairs of nodes
- There are no edges between nodes in same clause, so clique contains **one node from each clause**
- Set the corresponding **literals** to be **satisfied**
- Clique contains **no edges** between contradictory literals (i.e., no edge connects $x_i$ and $\overline{x_i}$ for any $i$)
- So, truth assignment is consistent and satisfies each clause (and the formula)

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## LAST STEP: SHOW **CLIQUE IS IN NP**

- YES-certificate: array of k nodes forming a clique
- Verify(I,C):
  - Check certificate is array of length k, containing vertex IDs
  - Check all-to-all edges to verify these vertices form a clique
  - $O(k^2) \subseteq O(|V|^2)$ runtime → polytime
  - **Correctness: exercise!** Need to prove:
    - if I is a yes instance, verify returns yes, and
    - if verify returns yes then I is a yes instance

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**Summary of Polynomial Transformations**

SAT ← Every problem in NP can be poly transformed to
↓
3-SAT
↓
Clique

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**Summary of Polynomial Transformations**

SAT ← Every problem in NP can be poly transformed to
↓
3-SAT
↓
Clique
↓
Vertex Cover

This additional poly transformation was proved last class (CL to VC)! **We also need to show Vertex Cover is in NP. Exercise.** ☺

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## REDUCING **VERTEX-COVER** TO **SUBSET-SUM**

(Proving Vertex-Cover $\leq_P$ Subset-Sum)

**(if we have time)**

## SUBSET-SUM (SLIGHTLY **DIFFERENT** FROM BEFORE)

**Problem 7.18**

**Subset Sum**

**Instance:** *A list of* **sizes** $S = [s_1, \ldots, s_n]$*; and a* **target sum**, $W$. *These are all positive integers.*

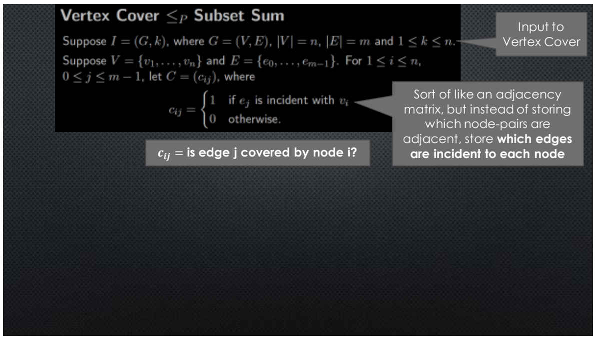
**Question:** *Does there exist a subset* $J \subseteq \{1, \ldots, n\}$ *such that* $\sum_{i \in J} s_i = W$?

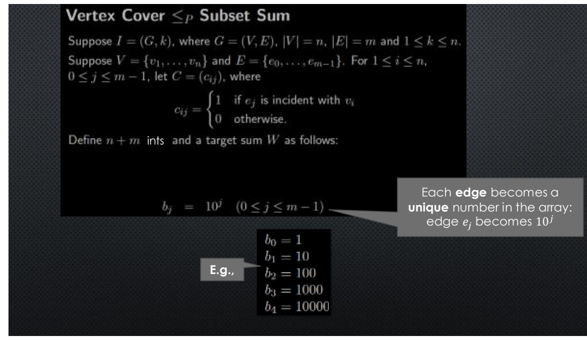
- Earlier, we defined Subset-Sum with a **target sum of 0**
- Here we add a **target sum T** and take **positive integers** as input

**Goal:** transform instance $I$ of VC into instance $f(I)$ of SS (in poly time) **such that** $I$ is a yes-instance of VC iff $f(I)$ is a yes-instance of SS

**Idea:** turn **nodes and edges** into a **list of integers** and a **target sum W.** Sum W should be achievable **IFF** there is a k-vertex cover.

Somehow want **the array of integers** to **encode** which edges are covered by various nodes, and **target sum** to **encode** that every edge is covered if W is achieved


Vertex Cover $\leq_P$ Subset Sum
Suppose $I = (G, k)$, where $G = (V, E)$, $|V| = n$, $|E| = m$ and $1 \leq k \leq n$.
Suppose $V = \{v_1, \ldots, v_n\}$ and $E = \{e_0, \ldots, e_{m-1}\}$. For $1 \leq i \leq n$, $0 \leq j \leq m-1$, let $C = (c_{ij})$, where
$c_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise.} \end{cases}$
Input to Vertex Cover
Sort of like an adjacency matrix, but instead of storing which node-pairs are adjacent, store **which edges** are **incident** to each node
$c_{ij}$ = is edge j covered by node i?


Vertex Cover $\leq_P$ Subset Sum
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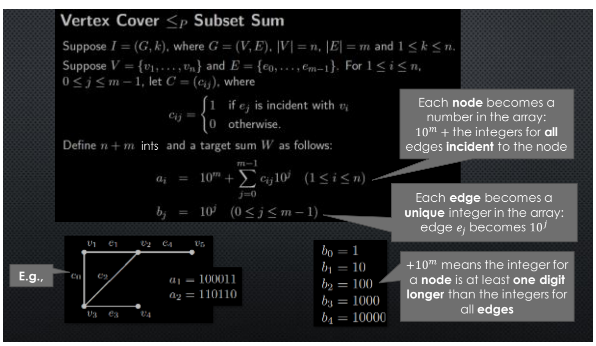
$c_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise.} \end{cases}$
Define $n + m$ ints and a target sum $W$ as follows:
$b_j = 10^j \quad (0 \leq j \leq m-1)$
Each **edge** becomes a **unique** number in the array: edge $e_j$ becomes $10^j$
E.g.,
$b_0 = 1$
$b_1 = 10$
$b_2 = 100$
$b_3 = 1000$
$b_4 = 10000$


Vertex Cover $\leq_P$ Subset Sum
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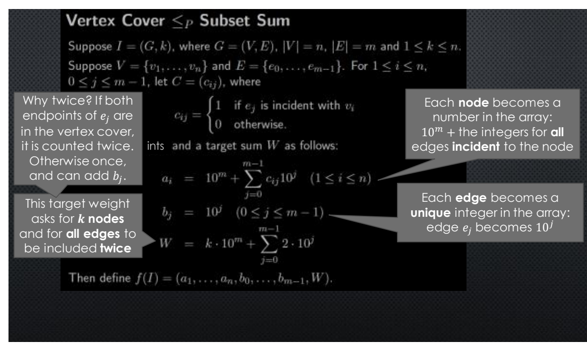
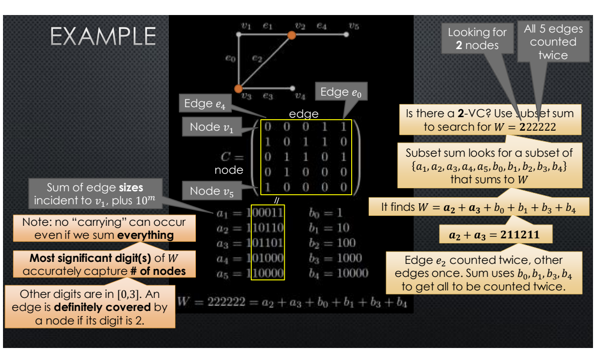
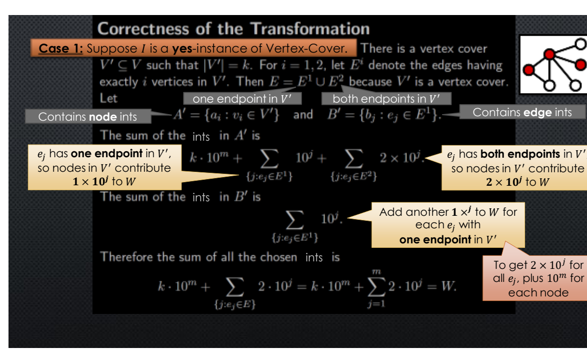
Suppose $V = \{v_1, \ldots, v_n\}$ and $E = \{e_0, \ldots, e_{m-1}\}$. For $1 \leq i \leq n$, $0 \leq j \leq m-1$, let $C = (c_{ij})$, where
$c_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise.} \end{cases}$
Define $n + m$ ints and a target sum $W$ as follows:
$a_i = 10^m + \sum_{j=0}^{m-1} c_{ij} 10^j \quad (1 \leq i \leq n)$
$b_j = 10^j \quad (0 \leq j \leq m-1)$
Each **node** becomes a number in the array: $10^m$ + the integers for **all** edges **incident** to the node
Each **edge** becomes a **unique** integer in the array: edge $e_j$ becomes $10^j$
E.g.,
$v_1$ $e_1$ $v_2$ $e_4$ $v_5$
$e_0$ $e_2$
$v_3$ $e_3$ $v_4$
$a_1 = 100011$
$a_2 = 110110$
$b_0 = 1$
$b_1 = 10$
$b_2 = 100$
$b_3 = 1000$
$b_4 = 10000$
$+10^m$ means the integer for a **node** is at least **one digit longer** than the integers for all **edges**


Vertex Cover $\leq_P$ Subset Sum
Suppose $I = (G, k)$, where $G = (V, E)$, $|V| = n$, $|E| = m$ and $1 \leq k \leq n$.
Suppose $V = \{v_1, \ldots, v_n\}$ and $E = \{e_0, \ldots, e_{m-1}\}$. For $1 \leq i \leq n$, $0 \leq j \leq m-1$, let $C = (c_{ij})$, where
$c_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise.} \end{cases}$
ints and a target sum $W$ as follows:
$a_i = 10^m + \sum_{j=0}^{m-1} c_{ij} 10^j \quad (1 \leq i \leq n)$
$b_j = 10^j \quad (0 \leq j \leq m-1)$
$W = k \cdot 10^m + \sum_{j=0}^{m-1} 2 \cdot 10^j$
Then define $f(I) = (a_1, \ldots, a_n, b_0, \ldots, b_{m-1}, W)$.
Why twice? If both endpoints of $e_j$ are in the vertex cover, it is counted twice. Otherwise once, and can add $b_j$.
This target weight asks for ***k* nodes** and for **all edges** to be included **twice**
Each **node** becomes a number in the array: $10^m$ + the integers for **all** edges **incident** to the node
Each **edge** becomes a **unique** integer in the array: edge $e_j$ becomes $10^j$


EXAMPLE
$v_1$ $e_1$ $v_2$ $e_4$ $v_5$
$e_0$ $e_2$
$v_3$ $e_3$ $v_4$
Edge $e_4$
Edge $e_0$
edge
Node $v_1$
$C =$
node
Node $v_5$
$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$
//
$a_1 = 100011$
$a_2 = 110110$
$a_3 = 101101$
$a_4 = 101000$
$a_5 = 110000$
$b_0 = 1$
$b_1 = 10$
$b_2 = 100$
$b_3 = 1000$
$b_4 = 10000$
$W = 222222 = a_2 + a_3 + b_0 + b_1 + b_3 + b_4$
Sum of edge **sizes** incident to $v_1$, plus $10^m$
Note: no “carrying” can occur even if we sum **everything**
**Most significant digit(s)** of $W$ accurately capture **# of nodes**
Other digits are in [0,3]. An edge is **definitely covered** by a node if its digit is 2.
Looking for **2** nodes
All 5 edges counted twice
Is there a **2**-VC? Use subset sum to search for $W = \mathbf{2}22222$
Subset sum looks for a subset of $\{a_1, a_2, a_3, a_4, a_5, b_0, b_1, b_2, b_3, b_4\}$ that sums to $W$
It finds $W = \boldsymbol{a_2} + \boldsymbol{a_3} + b_0 + b_1 + b_3 + b_4$
$\boldsymbol{a_2 + a_3 = 211211}$
Edge $e_2$ counted twice, other edges once. Sum uses $b_0, b_1, b_3, b_4$ to get all to be counted twice.


Correctness of the Transformation
**Case 1:** Suppose $I$ is a **yes**-instance of Vertex-Cover. There is a vertex cover $V' \subseteq V$ such that $|V'| = k$. For $i = 1, 2$, let $E^i$ denote the edges having exactly $i$ vertices in $V'$. Then $E = E^1 \cup E^2$ because $V'$ is a vertex cover.
Let
one endpoint in $V'$
both endpoints in $V'$
Contains **node** ints
$A' = \{a_i : v_i \in V'\}$ and $B' = \{b_j : e_j \in E^1\}$.
Contains **edge** ints
The sum of the ints in $A'$ is
$k \cdot 10^m + \sum_{\{j : e_j \in E^1\}} 10^j + \sum_{\{j : e_j \in E^2\}} 2 \times 10^j.$
$e_j$ has **one endpoint** in $V'$, so nodes in $V'$ contribute $\mathbf{1 \times 10^j}$ to $W$
$e_j$ has **both endpoints** in $V'$, so nodes in $V'$ contribute $\mathbf{2 \times 10^j}$ to $W$
The sum of the ints in $B'$ is
$\sum_{\{j : e_j \in E^1\}} 10^j.$
Add another $\mathbf{1} \times^{\boldsymbol{j}}$ to $W$ for each $e_j$ with **one endpoint** in $V'$
Therefore the sum of all the chosen ints is
$k \cdot 10^m + \sum_{\{j : e_j \in E\}} 2 \cdot 10^j = k \cdot 10^m + \sum_{j=1}^{m} 2 \cdot 10^j = W.$
To get $2 \times 10^j$ for all $e_j$, plus $10^m$ for each node

**Case 2:** Suppose $f(I)$ is a **yes**-instance of Subset Sum.

- We show $I$ is a **yes**-instance of Vertex-Cover
- Since $f(I)$ is a yes-instance, there exists $A' \cup B'$ that sums to $W$
  - where $A'$ contains node ints and $B'$ contains edge ints
- Define $V' = \{v_i : a_i \in A'\}$. We claim $V'$ is a vertex cover of size $k$.
  - We must have $|V'| = k$ for the coefficient of $10^m$ to be $k$ (no carrying)
  - Suppose (for contra.) $V'$ does **not** cover some edge $e_j = (u, v)$
  - Then the coefficient of $10^j$ is **zero** for every $a_i \in A'$
  - But the coefficient of $10^j$ is 2, so a subset of $B'$ must sum to $2 \times 10^j$
  - But this is impossible (so $e_j$ is covered, so all edges are covered)

**Vertex Cover $\leq_P$ Subset Sum**

**Complexity of the transformation: Easy! Included for your notes.**

Suppose $I = (G, k)$, where $G = (V, E)$, $|V| = n$, $|E| = m$ and $1 \leq k \leq n$.

Suppose $V = \{v_1, \ldots, v_n\}$ and $E = \{e_0, \ldots, e_{m-1}\}$. For $1 \leq i \leq n$, $0 \leq j \leq m-1$, let $C = (c_{ij})$, where

Assume adjacency matrix and unit cost model for simplicity

$$c_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Compute $C$ with trivial algorithm in $O(nm)$ time

Define $n + m$ sizes and a target sum $W$ as follows:

$$\begin{aligned} a_i &= 10^m + \sum_{j=0}^{m-1} c_{ij} 10^j \quad (1 \leq i \leq n) \\ b_j &= 10^j \quad (0 \leq j \leq m-1) \\ W &= k \cdot 10^m + \sum_{j=0}^{m-1} 2 \cdot 10^j \end{aligned}$$

Compute $a_i$ by visiting all incident edges. Trivial algorithm yields $O(m)$ time for each $a_i$, totaling $O(nm)$ over all $i$

Trivial to compute all $b_j$ in $O(m)$ time

Trivial to compute $W$ in $O(m)$ time

Then define $f(I) = (a_1, \ldots, a_n, b_0, \ldots, b_{m-1}, W)$.

Total $O(nm)$ time. This is polynomial in the input graph size!

**Summary of Polynomial Transformations**

CNF-SAT $\swarrow$ Every problem in NP can be poly transformed to

$\downarrow$ can be poly transformed to

3-CNF-SAT

$\downarrow$

Clique

$\downarrow$

Vertex Cover

$\swarrow$

Subset Sum

Technically need to also show SubsetSum with target T is in NP (exercise) to know it is in NPC

## IS 2-SAT ALSO HARD?

(IF WE HAVE TIME – VERY UNLIKELY)

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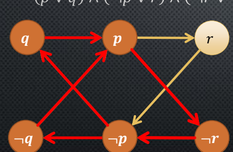
## 2-SAT EXAMPLES

- $(p \vee q) \wedge (\neg p \vee r) \wedge (\neg r \vee \neg p)$
  - Satisfiable: $p = 0, q = 1, r \in \{0,1\}$
- $(p \vee q) \wedge (\neg p \vee r) \wedge (\neg r \vee \neg p) \wedge (p \vee \neg q)$

Logical refresher: $p \Rightarrow q$ is **equivalent** to $\neg p \vee q$.

Therefore, $p \vee q$ is **equivalent** to $\neg p \Rightarrow q$ **and equivalent** to $\neg q \Rightarrow p$




Edges (implications of clauses)…

| | | | |
|---|---|---|---|
| $\neg p \Rightarrow q$ | $p \Rightarrow r$ | $r \Rightarrow \neg p$ | $\neg p \Rightarrow \neg q$ |
| $\neg q \Rightarrow p$ | $\neg r \Rightarrow \neg p$ | $p \Rightarrow \neg r$ | $q \Rightarrow p$ |

$q \Rightarrow p \Rightarrow \neg r \Rightarrow \neg p \Rightarrow \neg q$ … so $q$ cannot be $true$

$\neg q \Rightarrow p \Rightarrow \neg r \Rightarrow \neg p \Rightarrow q$ … so $q$ cannot be $false$

Therefore the formula **cannot** be satisfied!

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(variable names are integers in 1..|X|)

**2-SAT** can be solved in polynomial time. Suppose we are given an instance $I$ of **2-SAT** on a set of boolean variables $X = \{1..|X|\}$

(1) For every clause $x \vee y$ (where $x$ and $y$ are literals), construct two directed edges $\bar{x}y$ and $\bar{y}x$. We get a directed graph on vertex set $X \cup \bar{X}$.

(2) Determine the strongly connected components of this directed graph.

(3) $I$ is a yes-instance if and only if there is no strongly connected component containing $x$ and $\bar{x}$, for any $x \in X$.

Suppose no variable $x$ is in the same SCC as $\bar{x}$, then to get a satisfying assignment do the following:
For each $x$, if $\exists$ path from $x$ to $\bar{x}$, then set $x = false$ else set $x = true$.

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## BONUS SLIDES

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## SUMMARY OF COMPLEXITY CLASSES

See this slide's notes

- **P** (Poly-time) — E.g., (**decision** problem variants of:) BFS, Dijkstra's, **some** DP algorithms
  - **Decision** problems that can be solved by algorithms with runtime poly(input size)
- **NP** (Non-deterministic poly-time) — **All of P**, and e.g.,, vertex cover, clique, SAT, subset sum
  - **Decision** problems for which **certificates** can be **verified** in time poly(input size)
  - Equivalently: decision problems that can be solved in poly-time if you have access to a non-deterministic oracle that returns a yes-certificate if one exists
- **NPC** (NP-complete) — E.g., vertex cover, clique, SAT, subset sum, TSP-decision
  - **Decision** problems $\Pi \in NP$ s.t. every $\Pi' \in NP$ can be **transformed** to $\Pi$ in poly-time
- **NP-hard** (at least as hard as NPC) — **All of NPC**, and e.g., TSP-optimization, TSP-optimal value
  - problems $\Pi$ s.t. every $\Pi' \in NP$ can be **reduced** to $\Pi$ in poly-time

- Note: P, NP and NPC problems are **decidable**

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Found this neat image online