CS 341: ALGORITHMS

Lecture 4: divide & conquer III

Readings: see website

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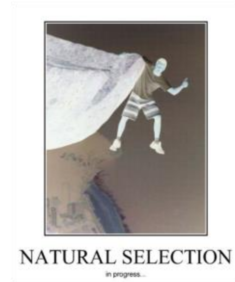
THE **SELECTION** PROBLEM



THE **SELECTION** PROBLEM

- Input: An array **A** containing **n** **distinct** integer values, and an integer **k** between 1 and **n**
- Output: The **k**-th **smallest** integer in **A**
- **Minimum** is a special case where $k = 1$
- **Median** is a special case where $k = \frac{n}{2}$
- **Maximum** is a special case where $k = n$
- Simple algorithm for solving selection?

Suppose we choose a pivot element y in the array A, and we restructure A so that all elements less than y precede y in A, and all elements greater than y occur after y in A. (This is exactly what is done in Quicksort, and it takes linear time.)

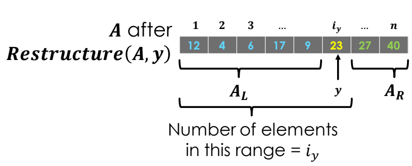
$y = 23$

A	12	4	6	27	23	17	40	9
	12	4	6	27	23	17	40	9
$Restructure(A, y)$	12	4	6	17	9	23	27	40

$y = 6$

A	12	4	6	27	23	17	40	9
	12	4	6	27	23	17	40	9
$Restructure(A, y)$	4	6	12	27	23	17	40	9

Number of elements on each side depend on the **value** y...



- **What's the k-th smallest element of A?**
  - If $k = i_y$ then y
  - If $k < i_y$ then the kth smallest in A_L
  - If $k > i_y$ then the $(k - i_y)$th smallest in A_R

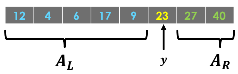
Recursive calls


```
QuickSelect(k, A[1..n])
    if n = 1 then return A[1]   // base case

    y = A[1]                    // pick an arbitrary pivot
    (AL, AR, iy) = Restructure(A, y)

    if k == iy return y
    else if k < iy then       return QuickSelect(k, AL)
    else /* k > iy */         return QuickSelect(k - iy, AR)

Restructure(A[1..n], y)
    AL = new array[1..n]        // allocate more than enough
    AR = new array[1..n]        // to avoid need for expansion
    nL = 0, nR = 0

    for i = 1 .. n
        if   A[i] < y then  AL[nL++] = A[i]
        else A[i] > y then  AR[nR++] = A[i]

    return (AL, AR, nL+1)       // nL+1 is the new index of y
```

Precondition: $1 \le k \le n$

OVERLY OPTIMISTIC ANALYSIS ☺

$\boldsymbol{A}$ after $\boldsymbol{Restructure}(\boldsymbol{A}, \boldsymbol{y})$



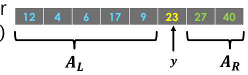
- If $i_y = \frac{n}{2}$, then we recurse on $\sim\frac{n}{2}$ elements,
- If we could **always** recurse on $\frac{n}{2}$ elements then
  - We would get $T(n) = T\left(\frac{n}{2}\right) + \Theta(n)$
  - Which would yield $a = 1, b = 2, y = 1, x = \log_2 1 = 0$, $y > x$ and $T(n) \in \Theta(n^y) = \boldsymbol{\Theta(n)}$ by the Master theorem.

But we **don't** always recurse on $\frac{n}{2}$ elements!

7

WORST-CASE ANALYSIS

$\boldsymbol{A}$ after $\boldsymbol{Restructure}(\boldsymbol{A}, \boldsymbol{y})$

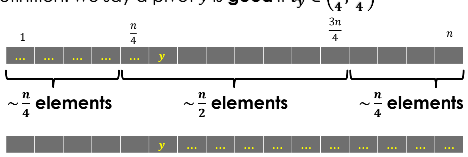


- If we always get $i_y = 1$ and recurse on the right, then
- We get $T(n) = T(n-1) + \Theta(n)$
- By the substitution method this is $\boldsymbol{\Theta(n^2)}$
- So, sometimes the pivot is good, sometimes it's bad…
- **What about the average case?**

8

AVERAGE-CASE ANALYSIS

- Definition: we say a pivot y is **good** if $\boldsymbol{i_y} \in \left(\frac{\boldsymbol{n}}{\boldsymbol{4}}, \frac{\boldsymbol{3n}}{\boldsymbol{4}}\right)$



- For **any good pivot** we recurse on at most $\frac{3n}{4}$ elements

Reducing the size of the subproblem by at least 1/4

- Probability of an arbitrary pivot being **good**?

p = 1/2

9

PROOF SKETCH

- Since probability of a good pivot is ½,
- on average, every **two** recursive calls, we will encounter a **good pivot**
- Encountering a good pivot reduces problem size to at most $\frac{3n}{4}$
- So, problem size is reduced to $\frac{3n}{4}$ after **expected linear work**
- **Average case recurrence:** $\boldsymbol{T(n) = T\left(\frac{3n}{4}\right) + \Theta(n)}$
  - $\boldsymbol{T(n) \in \Theta(n)}$

10

Here is a more rigorous proof of the average-case complexity: We say the algorithm is in phase j if the current subarray has size s, where

$$n\left(\frac{3}{4}\right)^{j+1} < s \le n\left(\frac{3}{4}\right)^{j}.$$

This is just for your notes, in case you want to know how you'd do this analysis formally

Let X_j be a random variable that denotes the amount of computation time occurring in phase j. If the pivot is in the middle half of the current subarray, then we transition from phase j to phase $j+1$. This occurs with probability $1/2$, so the expected number of recursive calls in phase j is 2. The computing time for each recursive call is linear in the size of the current subarray, so $E[X_j] \le 2cn(3/4)^j$ (where $E[\cdot]$ denotes the expectation of a random variable). The total time of the algorithm is given by $X = \sum_{j \ge 0} X_j$. Therefore

$$E[X] = \sum_{j \ge 0} E[X_j] \le 2cn \sum_{j \ge 0} (3/4)^j = 8cn \in O(n).$$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \text{ for } |r| < 1.$$

11

TAKING **SELECTION** FURTHER

- We just showed:
  - QuickSelect with **average case** runtime in $O(n)$
- Next up:
  - Median-of-medians QuickSelect (MOMQuickSelect)
  - **worst case** runtime in $O(n)$

Relies on getting a **good pivot** within $O(1)$ recursive calls **on average**

Must get a **good pivot** within $O(1)$ recursive calls **always**

The algorithm we will see picks a **good pivot** in **every** recursive call

12

HIGH LEVEL ALGORITHM

- Similar to QuickSelect
  - **Choose** a pivot
  - Move smaller elements to the left of the pivot, and larger elements to the right of the pivot
  - Recursively call MOMQuickSelect on one subarray
- Only difference is **how** we choose the pivot
  - **Always** want to pick a **good pivot**

13

ALWAYS PICKING A **GOOD PIVOT**

Example input A[1...50]:	11, 38, 6, 21, 20, 17, 14, 9, 7, 5, 8, 34, 49, 47, 28, 18, 44, 31, 46, 48, 27, 4, 2, 50, 23, 45, 3, 13, 43, 22, 10, 32, 35, 41, 24, 1, 30, 12, 15, 26, 16, 19, 36, 33, 37, 39, 25, 40, 29, 42

Group into **rows of 5**

11	38	6	21	20
17	14	9	7	5
8	34	49	47	28
18	44	31	46	48
27	4	2	50	23
45	3	13	43	22
10	32	35	41	24
1	30	12	15	26
16	19	36	33	37
39	25	40	29	42

Time complexity for this step?

Find **median** of **each row**

11	38	6	21	**20**
17	14	**9**	7	5
8	**34**	49	47	28
18	**44**	31	46	48
27	4	2	50	**23**
45	3	13	43	**22**
10	**32**	35	41	24
1	30	12	**15**	26
16	19	36	**33**	37
39	25	40	29	42

Time complexity for this step?

Build **array of medians**

20, 9, 34, 44, 23, 22, 32, 15, 33, 39

Time complexity?

Recursively find the median of these medians: **23**

Recursive problem size?

14

HOW GOOD IS THE PIVOT **23**?

elements ≤ 23 is **at least** 3(5). This is at least 3/10ths of our 50-element input, or $\mathbf{3n/10}$.

Recall: median of each row

11	38	6	21	**20**
17	14	**9**	7	5
8	**34**	49	47	28
18	**44**	31	46	48
27	4	2	50	***23***
45	3	13	43	**22**
10	**32**	35	41	24
1	30	12	**15**	26
16	19	36	**33**	37
39	25	40	29	42

Imagine sorting each row:

6	11	**20**
5	7	**9**
8	28	**34**
18	31	**44**
2	4	***23***
3	13	**22**
10	24	**32**
1	12	**15**
16	19	**33**
25	29	**39**

Then ordering rows by medians:

5	7	**9**	14	17
1	12	**15**	26	30
6	11	**20**	21	38
3	13	**22**	43	45
2	4	***23***	27	50
10	24	**32**	35	41
16	19	**33**	36	37
8	28	**34**	47	49
25	29	**39**	40	42
18	31	**44**	46	48

elements ≥ 23 is **at least** 3(6). This is > **3/10ths** of our 50-element input.

So, after restructuring, pivot 23 must have **at least** $\mathbf{3n/10}$ elements **before *and* after** it

This is a **good** pivot!

We recurse on A_L or A_R, and both have size **at most** $\mathbf{7n/10}$

15

$$\boldsymbol{MOMQuickSelect(k = 11, n = 14, A)}$$

11, 38, 6, 21, 20, 17, 14, 9, 7, 5, 8, 34, 49, 47

5, 6, 7, 9, 7, 11, 14, 17, 20, 21, $\boxed{34}$, 38, 47, 49

```
MOMQuickSelect(k, n, A)
    // base case
    if n <= 14 then sort(A) and return A[k]

    // divide and conquer to find medians
    r = (n-5) / 10
    medians[1..(2*r+1)] = new array
    for i = 1..(2*r+1)
        B[1..5] = A[(5*(i-1)+1)..(5*i)]
        sort(B)
        medians[i] = B[3]

    y = MOMQuickSelect(r+1, 2*r+1, medians)

    // divide and conquer to find rank k
    (AL, AR, iy) = Restructure(A, y)
    if        k == iy      then return y
    else if k < iy         then return MOMQuickSelect(k, iy-1, AL)
    else /* k > iy */      then return MOMQuickSelect(k-iy, n-iy, AR)
```

16

$$\boldsymbol{MOMQuickSelect(k = 11, n = 21, A)}$$

$\boxed{11, 38, 6, 21, 20, 17, 14, 9, 7, 5, 8, 34, 49, 47, 28}$, 18, 44, 31, 46, 48, 27

$$r = \left\lfloor \frac{21-5}{10} \right\rfloor = 1$$

Not considering at most 9 elements

$\boldsymbol{B}$	11	38	6	21	20
$\boldsymbol{sort(B)}$	6	11	**20**	21	38
	17	14	9	7	5
	5	7	**9**	14	17
	8	34	49	47	28
	8	28	**34**	47	49

$\boldsymbol{medians}$ 20, 9, 34

$$\boldsymbol{y = MOMQuickSelect(2, 3, [20, 9, 34]) \Rightarrow 20}$$

```
MOMQuickSelect(k, n, A)
    // base case
    if n <= 14 then sort(A) and return A[k]

    // divide and conquer to find medians
    r = (n-5) / 10
    medians[1..(2*r+1)] = new array
    for i = 1..(2*r+1)
        B[1..5] = A[(5*(i-1)+1)..(5*i)]
        sort(B)
        medians[i] = B[3]

    y = MOMQuickSelect(r+1, 2*r+1, medians)

    // divide and conquer to find rank k
    (AL, AR, iy) = Restructure(A, y)
    if        k == iy      then return y
    else if k < iy         then return MOMQuickSelect(k, iy-1, AL)
    else /* k > iy */      then return MOMQuickSelect(k-iy, n-iy, AR)
```

17

$$\boldsymbol{MOMQuickSelect(k = 11, n = 21, A)}$$

11, 38, 6, 21, 20, 17, 14, 9, 7, 5, 8, 34, 49, 47, 28, 18, 44, 31, 46, 48, 27

$$\boldsymbol{Restructure(A, y = 20) \Rightarrow}$$

$\boldsymbol{A_L} = [11, 6, 17, 14, 9, 7, 5, 8, 18]$

$\boldsymbol{A_R} = [38, 21, 34, 49, 47, 28, 44, 31, 46, 48, 27]$

$\boldsymbol{i_y} = |A_L| + 1 = \mathbf{10}$

$$\boldsymbol{k = 11 \quad > \quad i_y = 10}$$

$$\boldsymbol{k - i_y = 1 \qquad n - i_y = 10}$$

$$\boldsymbol{MOMQuickSelect(1, 10, A_R) \Rightarrow 21}$$

```
MOMQuickSelect(k, n, A)
    // base case
    if n <= 14 then sort(A) and return A[k]

    // divide and conquer to find medians
    r = (n-5) / 10
    medians[1..(2*r+1)] = new array
    for i = 1..(2*r+1)
        B[1..5] = A[(5*(i-1)+1)..(5*i)]
        sort(B)
        medians[i] = B[3]

    y = MOMQuickSelect(r+1, 2*r+1, medians)

    // divide and conquer to find rank k
    (AL, AR, iy) = Restructure(A, y)
    if        k == iy      then return y
    else if k < iy         then return MOMQuickSelect(k, iy-1, AL)
    else /* k > iy */      then return MOMQuickSelect(k-iy, n-iy, AR)
```

18

Runtime? (unit cost)

```
// base case
if n <= 14 then sort(A) and return A[k]

// divide and conquer to find medians
r = (n-5) / 10
medians[1..(2*r+1)] = new array
for i = 1..(2*r+1)
    B[1..5] = A[(5*(i-1)+1)..(5*i)]
    sort(B)
    medians[i] = B[3]

y = MOMQuickSelect(r+1, 2*r+1, medians)

// divide and conquer to find rank k
(AL, AR, iy) = Restructure(A, y)
if      k == iy      then return y
else if k < iy       then return MOMQuickSelect(k, iy-1, AL)
else /* k > iy */    then return MOMQuickSelect(k-iy, n-iy, AR)
```

$\Theta(1)$

$\Theta(n)$

$\Theta(n)$ iterations

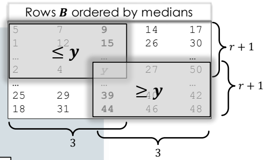
$\Theta(1)$

$T\left(\frac{n}{5}\right)$

$\Theta(n)$

$\Theta(1)$

$T(?)$



$3(r+1)$ elements $\leq y$
$3(r+1)$ elements $\geq y$

So problem size shrinks by at least $3(r+1)$

Observe $n = 10r + 5$

HOW MUCH DOES THE PROBLEM SHRINK?

- Shrinks by at least $3(r+1)$
- Problem size ~= $n = 10r + 5$
- Subproblem size $\leq n - Shrink = n - 3(r+1)$
  - $= 10r + 5 - 3r - 3 = 7r + 2$
  - Express in terms of n using $r = \left\lfloor \frac{n-5}{10} \right\rfloor$
    - Subproblem size $\leq 7\left\lfloor \frac{n-5}{10} \right\rfloor + 2 \leq 7\frac{n-5}{10} + 2$
    - $= \frac{7n}{10} - 7\left(\frac{5}{10}\right) + 2 = \frac{7n}{10} - \frac{3}{2} \leq \frac{\mathbf{7n}}{\mathbf{10}}$

Time complexity

```
// base case
if n <= 14 then sort(A) and return A[k]

// divide and conquer to find medians
r = (n-5) / 10
medians[1..(2*r+1)] = new array
for i = 1..(2*r+1)
    B[1..5] = A[(5*(i-1)+1)..(5*i)]
    sort(B)
    medians[i] = B[3]

y = MOMQuickSelect(r+1, 2*r+1, medians)

// divide and conquer to find rank k
(AL, AR, iy) = Restructure(A, y)
if      k == iy      then return y
else if k < iy       then return MOMQuickSelect(k, iy-1, AL)
else /* k > iy */    then return MOMQuickSelect(k-iy, n-iy, AR)
```

$\Theta(1)$

$\Theta(n)$

$\Theta(n)$ iterations

$\Theta(1)$

$T\left(\frac{n}{5}\right)$

$\Theta(n)$

$\Theta(1)$

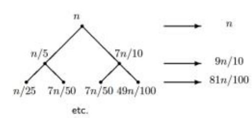
$T\left(\frac{\mathbf{7n}}{\mathbf{10}}\right)$

$T(n) \in O(n) + T(n/5) + T(7n/10)$	if $n \geq 15$
$T(n) \in O(1)$	if $n \leq 14$

The key fact is that $1/5 + 7/10 = 9/10 < 1$.

$T(n) \in O(n) + T(n/5) + T(7n/10)$	if $n \geq 15$
$T(n) \in O(1)$	if $n \leq 14$

The fact that $T(n) \in \Theta(n)$ can be proven formally using guess-and-check (induction) or informally using the recursion tree method.



$$\sum_{i=0}^{\infty} n\left(\frac{9}{10}\right)^i = 10n \in \Theta(n)$$

Guess & check
$\boldsymbol{T(n) = cn}$

- **Let $T(n) = c'n + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$ where $c' > 0$**
- **Want to prove: $T(n) = cn$ for some $c > 0$**
- Note c and c' are independent constants
  - c' comes from the work at each level of recursion being $O(n)$
  - c is a positive constant we are trying to show exists
- **I.H.:** Suppose $\exists c > 0 : T(n') = cn'$ for $15 \leq n' < n$
- $T(n) = c'n + c\frac{n}{5} + c\frac{7n}{10}$ (by inductive hypoth.)
- $\boldsymbol{T(n) = cn}$ **(want this to be true)**
- $\Leftrightarrow\ c'n + c\frac{n}{5} + c\frac{7n}{10} = cn$ (equivalently)
- $\Leftrightarrow c' + c\frac{1}{5} + c\frac{7}{10} = c \Leftrightarrow c = 10c'$ (by algebra)