# CS 341: ALGORITHMS

**Lecture 8: dynamic programming II**

Readings: see website


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# ROD CUTTING

## A "REAL" DYNAMIC PROGRAMMING EXAMPLE

- Input:
  - $n$: length of rod
  - $p_1, \dots, p_n$: $p_i$ = price of a rod of length $i$

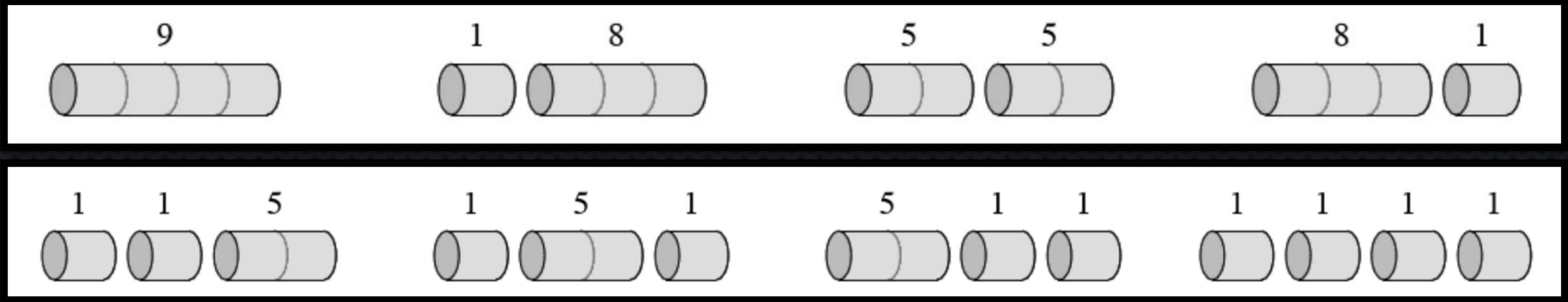
$n = 4$

| length $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| price $p_i$ | 1 | 5 | 8 | 9 |

- Output:
  - Max **income** possible by cutting the rod of length $n$ into any number of **integer** pieces (maybe **no** cuts)

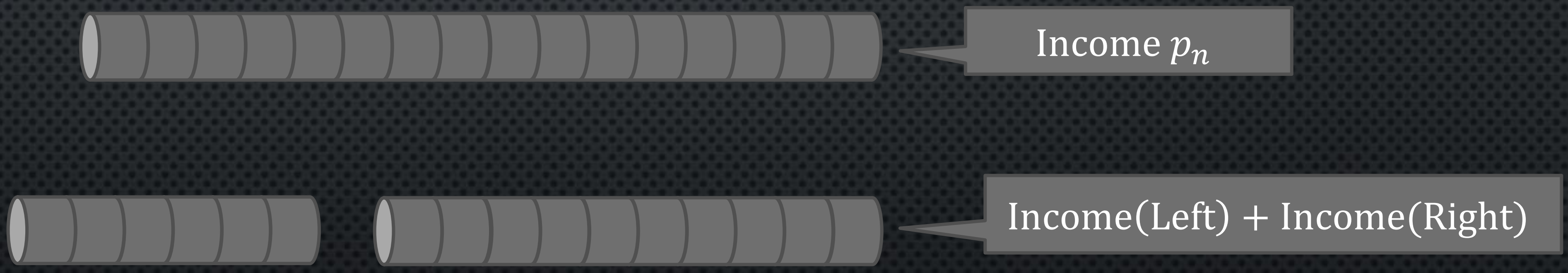
**All** ways of cutting a rod of length 4

**Example output: 10**



# DYNAMIC PROGRAMMING APPROACH

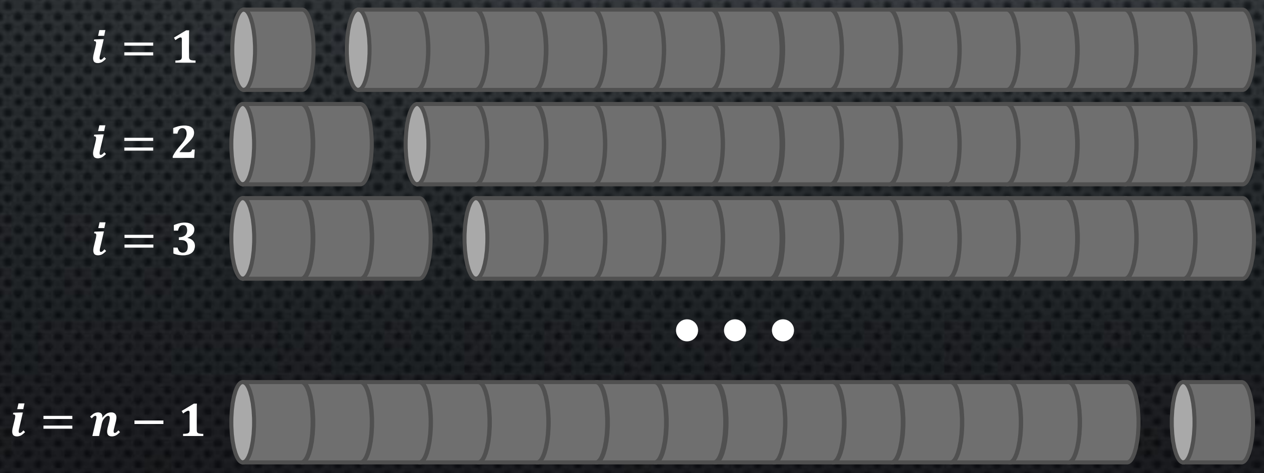
- High level idea **(can just think recursively to start)**
  - Given a rod of length n
  - Either make no cuts,
    or make a cut and **recurse** on the remaining parts




  - **Where** should we cut?

# DYNAMIC PROGRAMMING APPROACH

- Try **all ways** of making that cut
  - I.e., try a cut at positions $1, 2, \ldots, n-1$
  - In each case, recurse on two rods $[0, i]$ and $[i, n]$
- Take the max income over **all possibilities** (each $i$ / no cut)




Optimal substructure: Max income from two rods w/sizes $i$ and $n-i$

… is max income we can get from the rod size $i$

+ max income we can get from the rod size $n-i$

# RECURRENCE RELATION

Critical step! Must define what M(k) means, semantically!

- Define $M(k)$ = maximum income for rod of length $k$
- If we do **not** cut the rod, max income is $\boldsymbol{p_k}$
- If we **do** cut a rod **at** $\boldsymbol{i}$


  - max income is $M(i) + M(k-i)$
  - Want to maximize this **over all** $\boldsymbol{i}$
    - $\boldsymbol{max_i\{M(i) + M(k-i)\}}$ (for $0 < i < k$)
- $\boldsymbol{M(k) = \max\left\{p_k, \max_{1 \le i \le k-1}\{M(i) + M(k-i)\}\right\}}$

# COMPUTING SOLUTIONS BOTTOM-UP

- **Recurrence:** $M(k) = max\{p_k, \max_{1 \le i \le k-1}\{M(i) + M(k-i)\}\}$
- Compute **table** of solutions: $\mathrm{M}[1..n]$


- Dependencies: **entry** $k$ depends on
  - $M[i]$ → $M[1..(k-1)]$
  - $M[k-i]$ → $M[1..(k-1)]$
  - All of these dependencies are $< k$
- So we can fill in the table entries in order $1..n$

Recall, semantically, $M(k)$ = maximum income for rod of length $k$

**Recurrence:** $\boldsymbol{M(k) = max\{p_k, \max_{1 \le i \le k-1}\{M(i) + M(k-i)\}\}}$

```
RodCutting(n, p[1..n])
    M = new array[1..n]

    // compute each entry M[k]
    for k = 1..n
        M[k] = p[k] // current best = no cuts

        // try each cut in 1..(k-1)
        for i = 1..(k-1)
            M[k] = max(M[k], M[i] + M[k-i])

    return M[n]
```

| Time complexity (unit cost)? | $\Theta(n^2)$ |
|---|---|

# MISCELLANEOUS TIPS

- Building a table of results bottom-up is what makes an algorithm DP
- There is a similar concept called **memoization**
  - But, for the purposes of this course, we want to see bottom-up table filling!
- Base cases are **critical**
  - They often completely determine the answer
  - Try setting f[0]=f[1]=0 in FibDP…

# **DP** SOLUTION TO **0-1 KNAPSACK**

Suppose the optimal solution **O does not include this**

Then with the **O** must achieve the best possible value **using only items 1-3**.


Problem: output maximum value one can get from taking **≤ 7kg**, out of these **four items**.

**Subproblem:** output max value for **≤ 7kg** out of these **three items**

This is a **smaller subproblem:** reduced # of items

Goal: create **recurrence relation** to describe optimal solution in terms of subproblems

Let P[i, m] = maximum profit using any subset of the items **1..i**, with weight limit **m**

Note: P[n, M] (= P[4, 7]) is the **optimal profit**

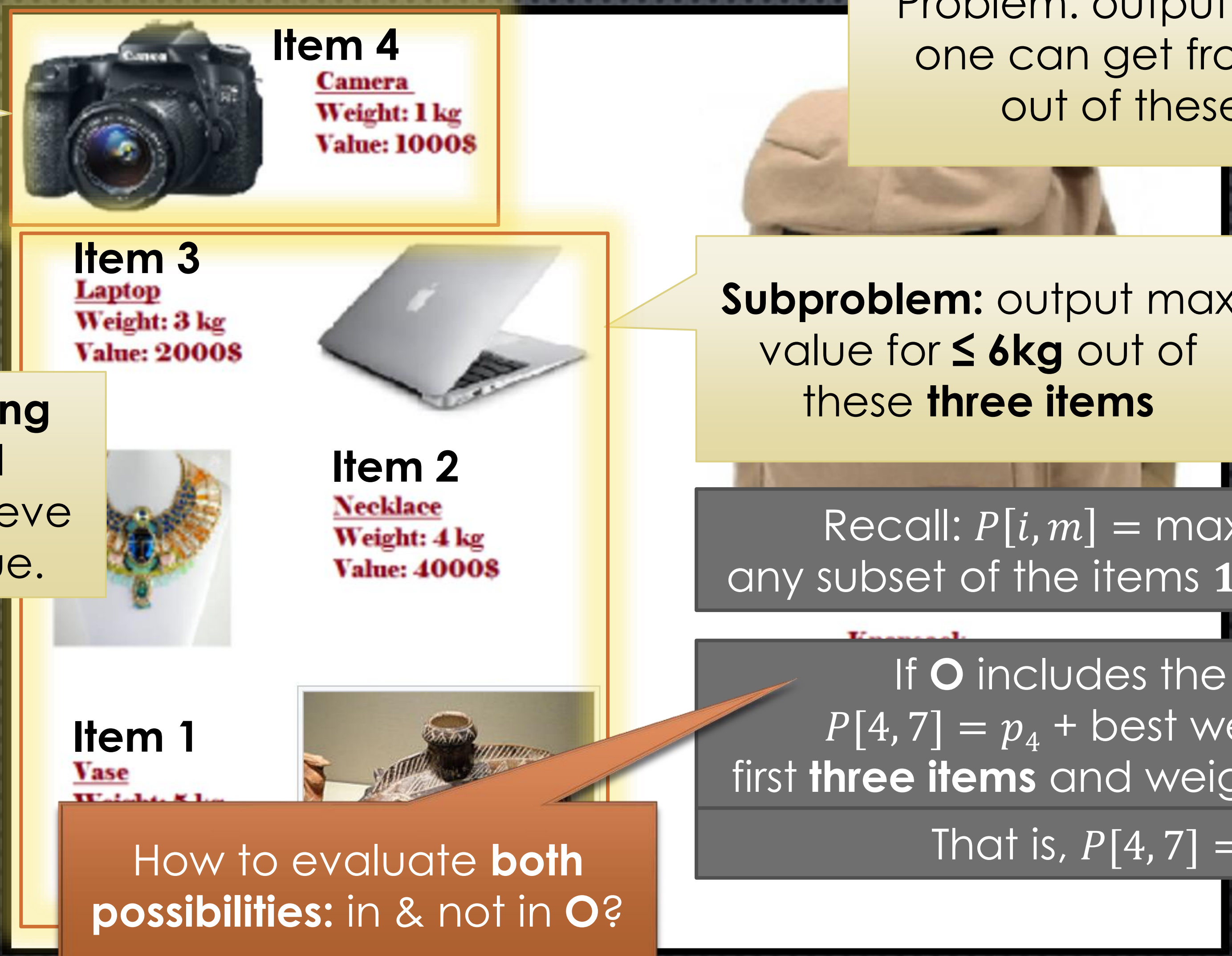
If **O does not include** the camera, then P[4, 7] = best we can do with the **first three** items and weight limit **7kg**

That is, P[4, 7] = P[3, 7]

**What if the camera IS included in O?**

Problem: output maximum value one can get from taking **≤ 7kg**, out of these **four items**.

Suppose the optimal solution **O includes this**




Then with the **remaining 7kg – $w_4$ = 6kg, and items 1-3, O** must achieve the best possible value.

**Subproblem:** output max value for **≤ 6kg** out of these **three items**

This is a **smaller subproblem:** reduced weight **and** # of items

Recall: $P[i, m]$ = maximum profit using any subset of the items $\mathbf{1..}\boldsymbol{i}$, with weight limit $\boldsymbol{m}$

If **O** includes the camera, then
$P[4, 7] = p_4$ + best we can do with the
first **three items** and weight limit **7kg – $w_4$** = 6kg

That is, $P[4, 7] = p_4 + P[3, 6]$

How to evaluate **both possibilities:** in & not in **O**?

Recall: $P[i, m]$ = maximum profit using any subset of the items $\mathbf{1..}\boldsymbol{i}$, with weight limit $\boldsymbol{m}$

**In general:**

| | | |
|---|---|---|
| If **O does not include** the camera, then $P[4,7]$ = best we can do with the **first three** items and weight limit **7kg** | $P[4,7] = P[3,7]$ | $P[i,m] = P[i-1,m]$ |
| If **O includes** the camera, then $P[4,7]$ = $\boldsymbol{p_4}$ + best we can do with the **first three** items and weight limit **7kg – w$_4$** = 6kg | $P[4,7] = p_4 + P[3, 7-w_4]$ | $P[i,m] = p_i + P[i-1, m-w_i]$ |
| **Try both** and take the better result! **(How?)** | $P[4,7] = \mathbf{max}\{P[3,7], p_4 + P[3, 7-w_4]\}$ | $P[i,m] = \mathbf{max}\{P[i-1,m], p_i + P[i-1, m-w_i]\}$ |

Note that $\max\{P[i-1,m], p_i + P[i-1,m-w_i]\}$ **is only valid if** $i \geq 2$ and $m \geq w_i$

What to do when $i = 1$ or $m < w_i$? These are **special cases.**

**General case:** $i \geq 2$ and $m \geq w_i$

Since $m \geq w_i$, we **can carry item i.**
$P[i, m] = \max\{P[i-1, m], p_i + P[i-1, m-w_i]\}$

**Special case 1:** $i \geq 2$ and $m < w_i$

Since $m < w_i$, we **cannot carry item i.**
So, $P[i, m] = P[i-1, m]$.

**Special case 2:** $i = 1$ and $m \geq w_i$

Since $i \leq 1$, we **can only use item 1.**
Since $m \geq w_i$, we **can carry item 1.**
So, $P[i, m] = p_i$.

**Special case 3:** $i = 1$ and $m < w_i$

Since $i \leq 1$, we **can only use item 1.**
Since $m < w_i$, we **cannot carry item 1.**
So, $P[i, m] = 0$.

**Recurrence Relation:**

$$P[i, m] = \begin{cases} \max\{P[i-1, m], p_i + P[i-1, m-w_i]\} & \text{if } i \geq 2,\ m \geq w_i \\ P[i-1, m] & \text{if } i \geq 2,\ m < w_i \\ p_1 & \text{if } i = 1,\ m \geq w_1 \\ 0 & \text{if } i = 1,\ m < w_1. \end{cases}$$

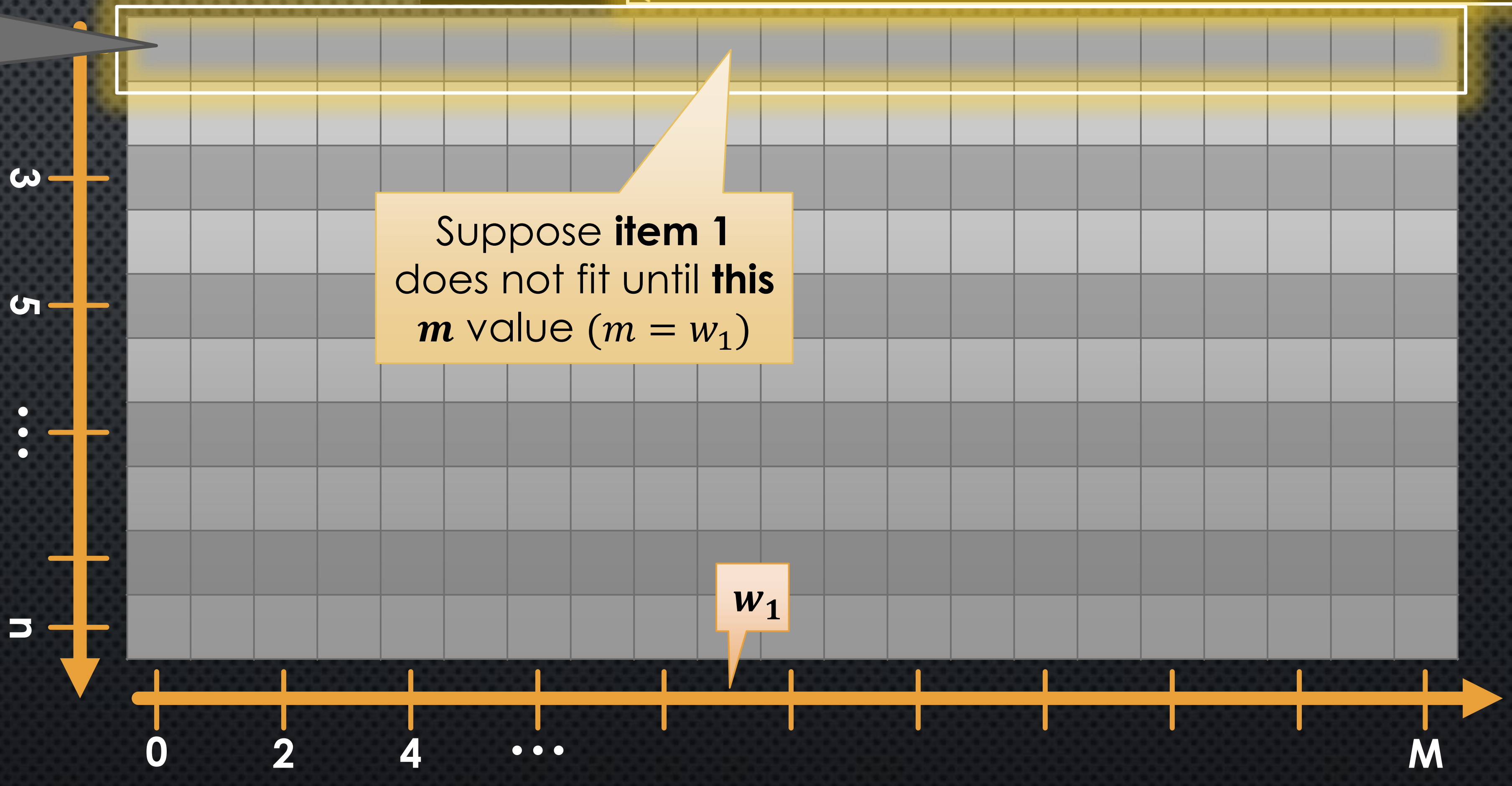
# FILLING THE ARRAY:

$$P[i,m] = \begin{cases} \max\{P[i-1,m], p_i + P[i-1, m-w_i]\} & \text{if } i \geq 2,\ m \geq w_i \\ P[i-1,m] & \text{if } i \geq 2,\ m < w_i \\ \boxed{p_1 \quad \text{if } i = 1,\ m \geq w_1} \\ \boxed{0 \quad \text{if } i = 1,\ m < w_1} \end{cases}$$

No data dependencies on any other array cells.

$i$-axis

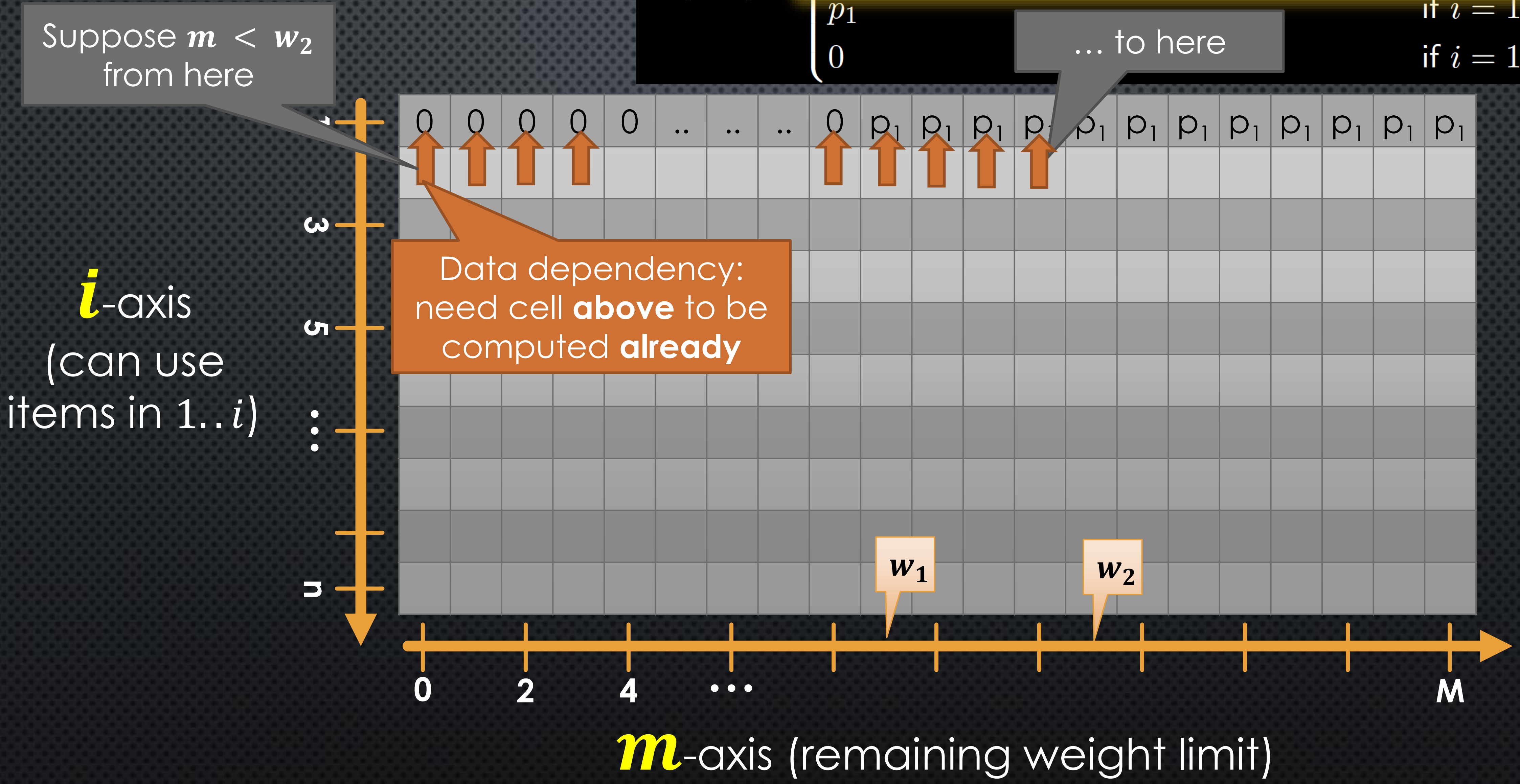
(can use items in $1..i$)




$m$-axis (remaining weight limit)

# FILLING THE ARRAY:

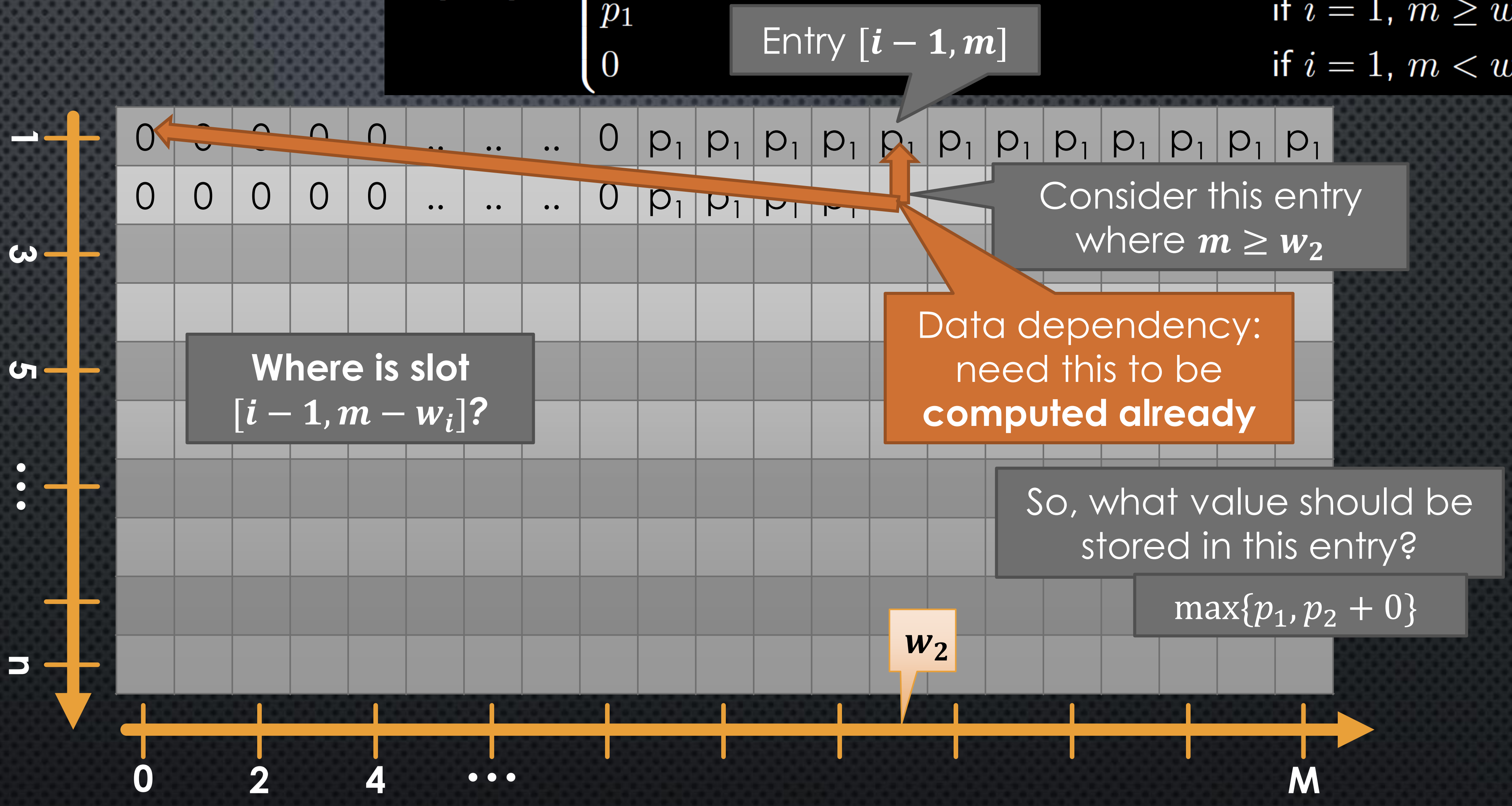
$$P[i,m] = \begin{cases} \max\{P[i-1,m], p_i + P[i-1,m-w_i]\} & \text{if } i \geq 2,\ m \geq w_i \\ \boxed{P[i-1,m] \quad \text{if } i \geq 2,\ m < w_i} \\ p_1 & \text{if } i = 1,\ m \geq w_1 \\ 0 & \text{if } i = 1,\ m < w_1. \end{cases}$$



# FILLING THE ARRAY:

$$P[i,m] = \begin{cases} \max\{P[i-1,m], p_i + P[i-1,m-w_i]\} & \text{if } i \ge 2,\ m \ge w_i \\ P[i-1,m] & \text{if } i \ge 2,\ m < w_i \\ p_1 & \text{if } i = 1,\ m \ge w_1 \\ 0 & \text{if } i = 1,\ m < w_1. \end{cases}$$

$i$-axis (can use items in $1..i$)

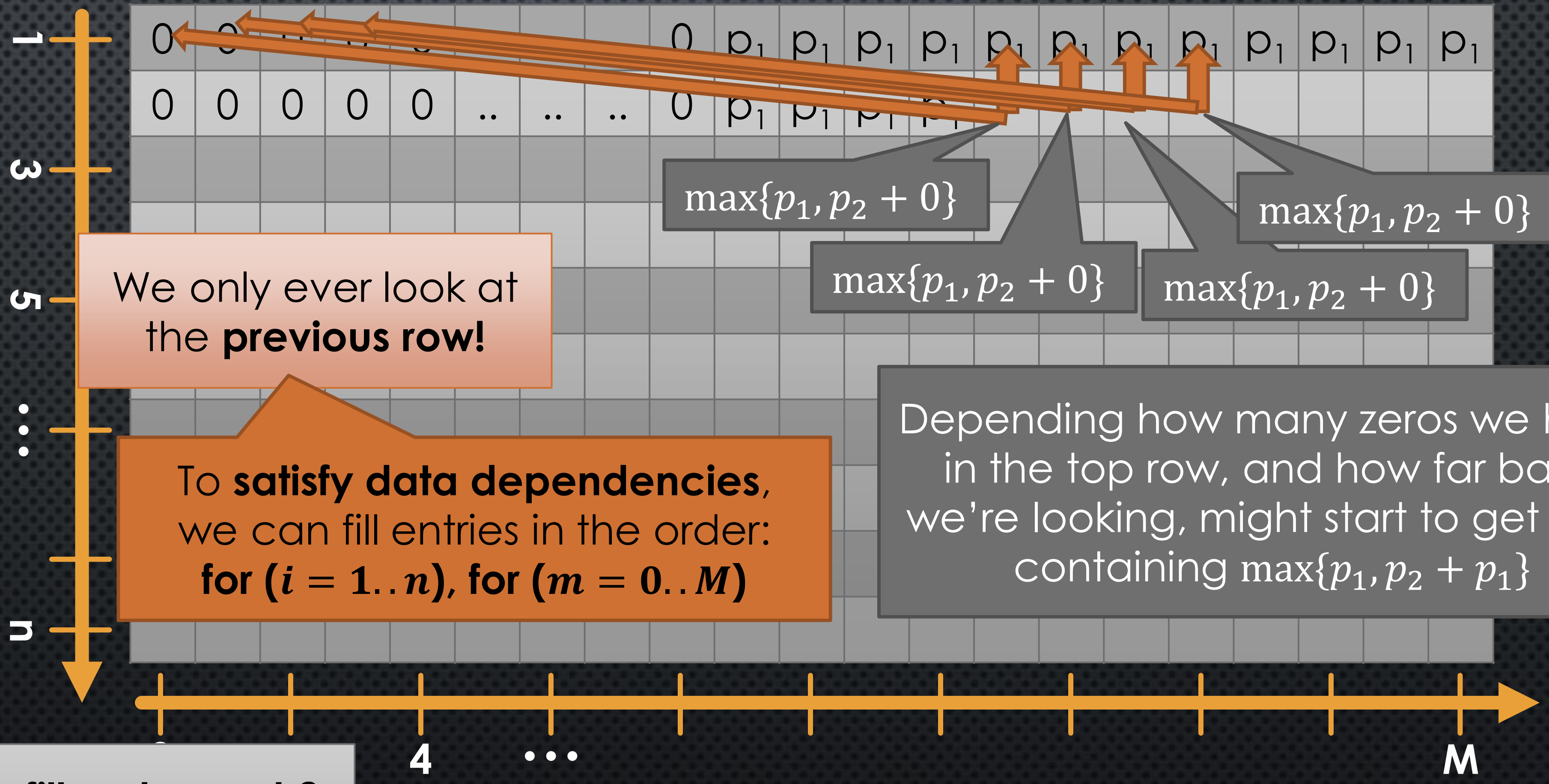



$m$-axis (remaining weight limit)

# FILLING THE ARRAY:

$$P[i,m] = \begin{cases} \boxed{\max\{P[i-1,m], p_i + P[i-1,m-w_i]\} \quad \text{if } i \geq 2,\ m \geq w_i} \\ P[i-1,m] & \text{if } i \geq 2,\ m < w_i \\ p_1 & \text{if } i = 1,\ m \geq w_1 \\ 0 & \text{if } i = 1,\ m < w_1. \end{cases}$$

$i$-axis (can use items in $1..i$)




$\max\{p_1, p_2 + 0\}$

$\max\{p_1, p_2 + 0\}$

$\max\{p_1, p_2 + 0\}$

$\max\{p_1, p_2 + 0\}$

We only ever look at the **previous row!**

To **satisfy data dependencies**, we can fill entries in the order: **for ($i = 1..n$), for ($m = 0..M$)**

Depending how many zeros we have in the top row, and how far back we're looking, might start to get cells containing $\max\{p_1, p_2 + p_1\}$

**Would the following fill-order work?**
for ($i = 1..n$), for ($m = M..0$)

$m$-axis (remaining weight limit)

# EXERCISE

$$\begin{cases} \max\{P[i-1,m], p_i + P[i-1, m-w_i]\} & \text{if } i \geq 2,\ m \geq w_i \\ P[i-1,m] & \text{if } i \geq 2,\ m < w_i \end{cases}$$

Suppose we have profits $1, 2, 3, 5, 7, 10$, weights $2, 3, 5, 8, 13, 16$, and capacity $30$.

The following table is computed:

$m$-axis (weight)

$i$-axis (items)

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | |
| 2 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | | | | | | | | | | | | | | |
| 3 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | ? | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

$P[3, 16] =$ **? What do you think ?**

# EXERCISE

$$\max\{P[i-1,m], p_i + P[i-1,m-w_i]\} \quad \text{if } i \geq 2,\ m \geq w_i$$
$$P[i-1,m] \quad \text{if } i \geq 2,\ m < w_i$$

Suppose we have profits $1, 2, 3, 5, 7, 10$, weights $2, 3, 5, 8, 13, 16$, and capacity $30$.

The following table is computed:

***m*-axis (weight)**

***i*-axis (items)**

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 4 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 8 | 8 | 9 | 10 | 10 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 |
| 5 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 8 | 8 | 9 | 10 | 10 | 11 | 11 | 11 | 12 | 12 | 13 | 14 | 14 | 15 | 15 | 16 | 17 | 17 |
| 6 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 18 |

$$P[3,16] = \max\{P[2,16], P[2,11] + 3\} = \max\{3, 3+3\} = 6.$$

**Recall:** To satisfy data dependencies, we can fill entries in the order: **for ($i = 1..n$), for ($m = 0..M$)**

$$P[i,m] = \begin{cases} \max\{P[i-1,m], p_i + P[i-1, m-w_i]\} & \text{if } i \geq 2,\ m \geq w_i \\ P[i-1,m] & \text{if } i \geq 2,\ m < w_i \\ p_1 & \text{if } i = 1,\ m \geq w_1 \\ 0 & \text{if } i = 1,\ m < w_1. \end{cases}$$

```
Knapsack01(p[1..n], w[1..n], M)
    P = new table[1..n][0..M]

    // base cases where i=1
    for m = 0..M
        if m < w[1] then
            P[1][m] = 0
        else
            P[1][m] = p[1]

    // general cases where i>=2
    for i = 2..n
        for m = 0..M
            if m < w[i] then
                P[i][m] = P[i-1][m]
            else
                P[i][m] = max(P[i-1][m],
                              p[i] + P[i-1][m-w[i]])

    return P[n][M]
```

Read & return optimal **profit**

How about the optimal **items?**

# OUTPUTTING **CONTENTS OF** THE OPTIMAL KNAPSACK **O**

The optimal solution is computed by tracing back through the table.

For the previous example, consisting of profits $1, 2, 3, 5, 7, 10$, weights $2, 3, 5, 8, 13, 16$, and capacity $30$, the optimal solution is **? ? ?**

**weight limit remaining**

| Items you can take | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| $\boxed{4}$ | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 8 | 8 | 9 | 10 | 10 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 |
| 5 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | [illegible] | 8 | 9 | 10 | 10 | 11 | 11 | 11 | 12 | 12 | 13 | 14 | 14 | 15 | 15 | 16 | 17 | 17 |
| $\boxed{6}$ | – | – | – | – | – | – | – | – | – | – | – | – | [illegible] | – | – | [illegible] | – | – | – | – | – | – | – | – | – | – | – | [illegible] | – | – | 18 |

8 > 6 so O **must** take item 4

**Same profit** using items 1..4 or 1..5. So, **there exists** an optimal solution O that does **not** use **item 5!** Consider O.

Best profit for remaining items + weight

18 > 17, so **any** optimal solution **must** take item 6

remaining weight = 14

Start at optimal profit

**Exercise:** continue, and determine which other items **are in O**

# OUTPUTTING **CONTENTS OF** THE OPTIMAL KNAPSACK **O**

The optimal solution is computed by tracing back through the table.

For the previous example, consisting of profits $1, 2, 3, 5, 7, 10$, weights $2, 3, 5, 8, 13, 16$, and capacity $30$, the optimal solution is $[1, 1, 0, 1, 0, 1]$.

weight limit remaining

| Items you can take | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 4 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 8 | 8 | 9 | 10 | 10 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 |
| 5 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 8 | 8 | 9 | 10 | 10 | 11 | 11 | 11 | 12 | 12 | 13 | 14 | 14 | 15 | 15 | 16 | 17 | 17 |
| 6 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 18 |

```
Knapsack01_Items(p[1..n], w[1..n], M, P)
    x = new array[1..n]
    i = n
    m = M

    while i > 1
        if P[i][m] == P[i-1][m]
            x[i] = 0
            i = i - 1
        else
            x[i] = 1
            m = m - w[i]
            i = i - 1

    x[1] = (P[i][m] > 0) ? 1 : 0
    return x
```

Runtime given $P$?

$\Theta(n)$

Is this linear time?

More on this soon…

# Complexity of the Algorithm

Suppose we assume the unit cost model, so additions / subtractions take time $O(1)$.

The complexity to construct the table is $\Theta(nM)$

Is this a polynomial-time algorithm, as a function of the size of the problem instance?

We have

$$\text{size}(I) = \log_2 M + \sum_{i=1}^{n} \log_2 w_i + \sum_{i=1}^{n} \log_2 p_i.$$

Note in particular that $M$ is exponentially large compared to $\log_2 M$. So constructing the table is not a polynomial-time algorithm, even in the unit cost model.

What would the complexity of a recursive algorithm be?

So the DP alg is faster when there are **many item types**, but **small weight limit**

Huge **n** is fine, but **M** should be in **poly(n)** to get an asymptotic improvement

DP takes $\boldsymbol{\Theta(nM)}$ **time**, which *could be* $\boldsymbol{\Theta(n2^n)}$ **for huge M**

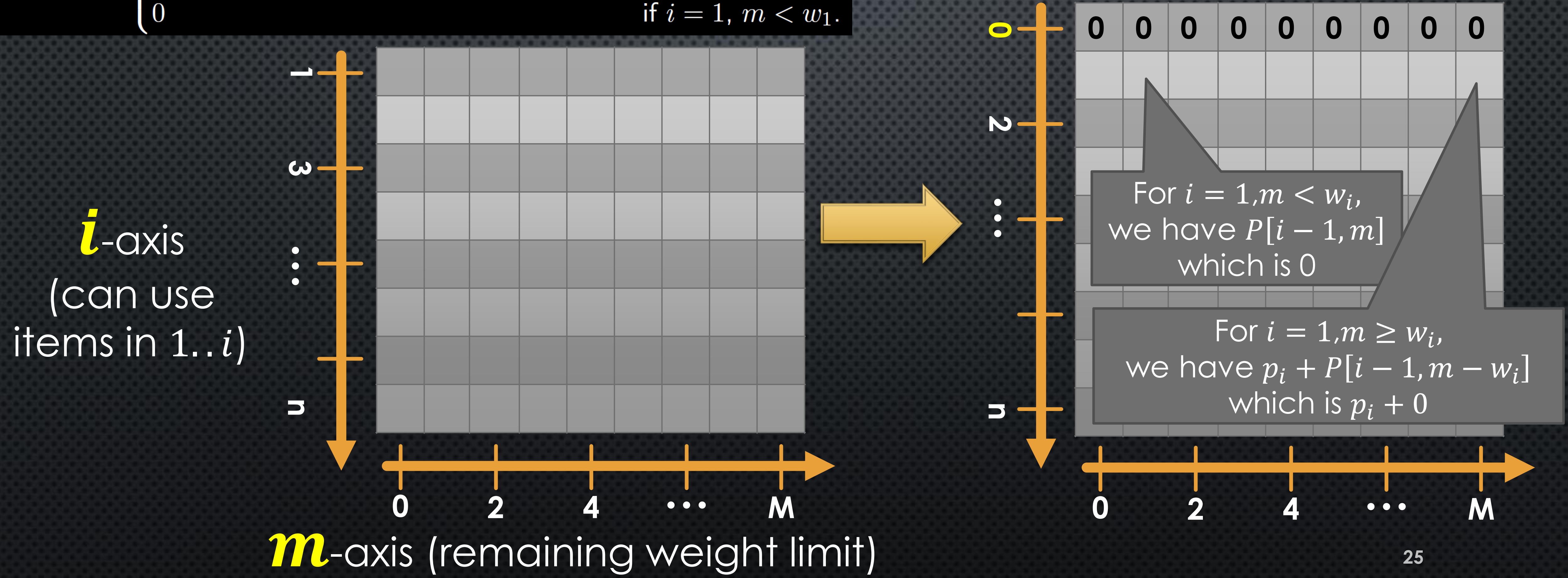
**n** must be **very small**

A recursive algorithm would take $\sim\boldsymbol{\Theta(2^n)}$ **time**

# SIMPLIFYING BASE CASES

$$P[i,m] = \begin{cases} \max\{P[i-1,m], p_i + P[i-1,m-w_i]\} & \text{if } i \geq 1, m \geq w_i \\ P[i-1,m] & \text{if } i \geq 1, m < w_i \\ 0 & \text{if } i = 0 \end{cases}$$

$$P[i,m] = \begin{cases} \max\{P[i-1,m], p_i + P[i-1,m-w_i]\} & \text{if } i \geq 2,\ m \geq w_i \\ P[i-1,m] & \text{if } i \geq 2,\ m < w_i \\ p_1 & \text{if } i = 1,\ m \geq w_1 \\ 0 & \text{if } i = 1,\ m < w_1. \end{cases}$$



```
Knapsack01(p[1..n], w[1..n], M)
    P = new table[0..n][0..M] containing zeros

    for i = 1..n
        for m = 0..M
            if m < w[i] then
                P[i][m] = P[i-1][m]
            else
                P[i][m] = max(P[i-1][m],
                              p[i] + P[i-1][m-w[i]])

    return P[n][M]
```

We get much simpler code!

Compare:

```
Knapsack01(p[1..n], w[1..n], M)
    P = new table[1..n][0..M]

    // base cases where i=1
    for m = 0..M
        if m < w[1] then
            P[1][m] = 0
        else
            P[1][m] = p[1]

    // general cases where i>=2
    for i = 2..n
        for m = 0..M
            if m < w[i] then
                P[i][m] = P[i-1][m]
            else
                P[i][m] = max(P[i-1][m],
                              p[i] + P[i-1][m-w[i]])

    return P[n][M]
```

# SAVING SPACE

```
Knapsack01(p[1..n], w[1..n], M)
    P = new table[0..n][0..M] containing zeros

    for i = 1..n
        for m = 0..M
            if m < w[i] then
                P[i][m] = P[i-1][m]
            else
                P[i][m] = max(P[i-1][m],
                              p[i] + P[i-1][m-w[i]])
```

```
Knapsack01(p[1..n], w[1..n], M)
    Pprev = new array[0..M] containing zeros
    P = new array[0..M] containing zeros

    for i = 1..n
        swap P and Pprev
        for m = 0..M
            if m < w[i] then
                P[m] = Pprev[m]
            else
                P[m] = max(Pprev[m], p[i] + Pprev[m-w[i]])

    return P[M]
```

We never look at P[i-2][…]. Just keep two arrays representing P[i] and P[i-1]

Space complexity changes from $O(mn)$ to $O(m)$


# COIN CHANGING

# Coin Changing

There **is** a denomination with **unit value**!

**Problem 5.2**

**Coin Changing**

**Instance:** *A list of* **coin denominations**, $1 = d_1, d_2, \ldots, d_n$, *and a positive integer* $T$, *which is called the* **target sum**.

**Find:** *An* $n$*-tuple of non-negative integers, say* $A = [a_1, \ldots, a_n]$, *such that* $T = \sum_{i=1}^{n} a_i d_i$ *and such that* $N = \sum_{i=1}^{n} a_i$ *is minimized.*

What subproblems should be considered?

What table of values should we fill in?

In 0-1 knapsack, we only considered **two subproblems** in our recurrence: **taking an item, or not.**

Here we can do **more than** *use a coin denomination or not*.

Let $N[i, t]$ denote the optimal solution to the subproblem consisting of the first $i$ coin denominations $d_1, \ldots, d_i$ and target sum $t$.

Exploring: some sensible base case(s)?

General case:
What are the different ways we could use coin denomination $\boldsymbol{d_i}$?
What subproblems / solutions should we use?

Final recurrence relation

Let $N[i,t]$ denote the optimal solution to the subproblem consisting of the first $i$ coin denominations $d_1, \ldots, d_i$ and target sum $t$.

Also $\boldsymbol{N[i,0] = 0}$ for all $i$

Since $d_1 = 1$, we immediately have $N[1,t] = t$ for all $t$.

General case:
What are the different ways we could use coin denomination $\boldsymbol{d_i}$?
What subproblems / solutions should we use?

Final recurrence relation

Let $N[i,t]$ denote the optimal solution to the subproblem consisting of the first $i$ coin denominations $d_1, \ldots, d_i$ and target sum $t$.

Also $\boldsymbol{N[i,0] = 0}$ for all $i$

Since $d_1 = 1$, we immediately have $N[1,t] = t$ for all $t$.

For $i \geq 2$, the number of coins of denomination $d_i$ is an integer $j$ where $0 \leq j \leq \lfloor t/d_i \rfloor$.

If we use $j$ coins of denomination $d_i$, then the target sum is reduced to $t - jd_i$, which we must achieve using the first $i-1$ coin denominations.

Thus we have the following recurrence relation:

$$N[i,t] = \begin{cases} \min\{j + N[i-1, t-jd_i] : 0 \leq j \leq \lfloor t/d_i \rfloor\} & \text{if } i \geq 2 \\ t & \text{if } i = 1 \text{ OR } t = 0 \end{cases}$$

# FILLING THE ARRAY

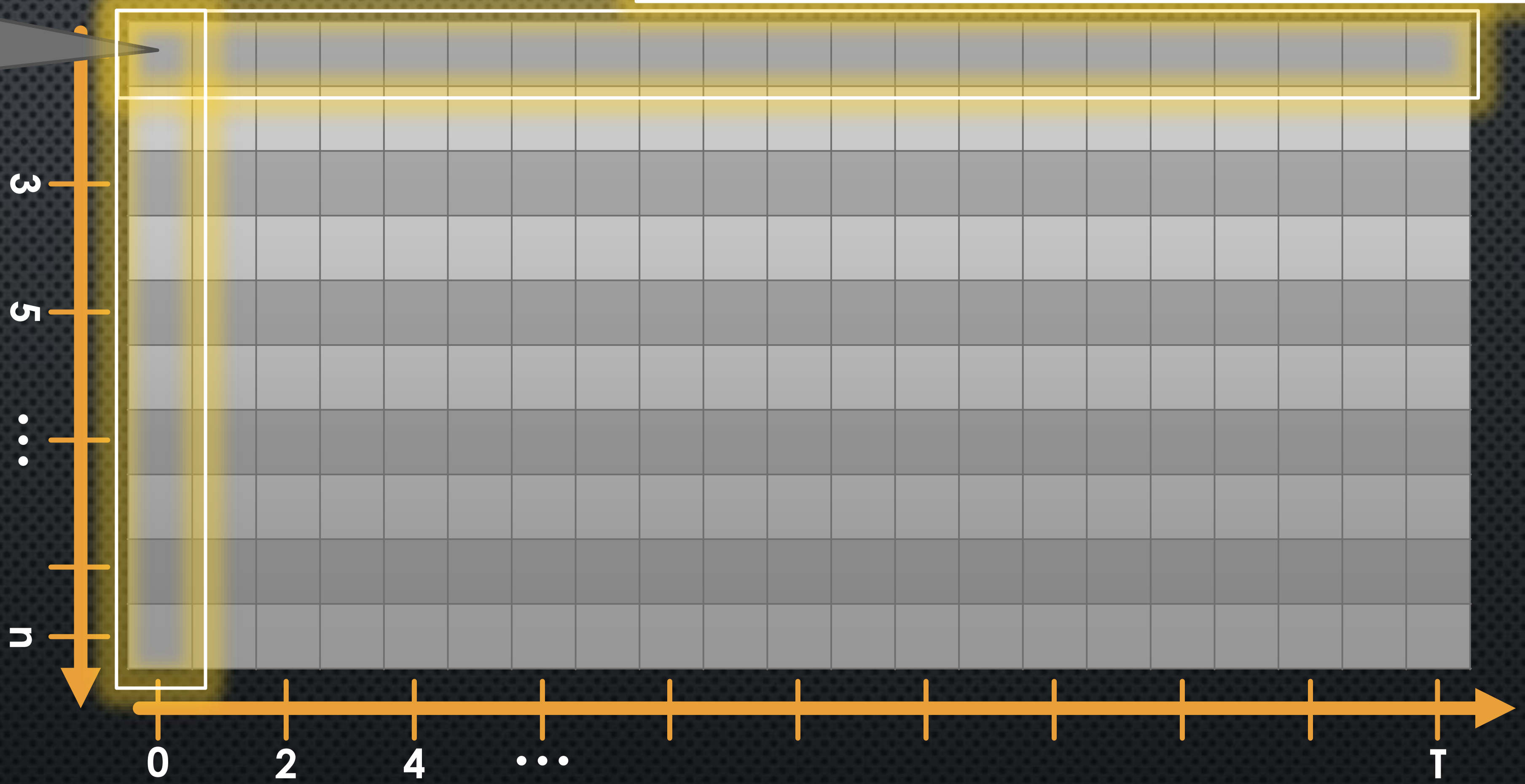
$$N[i,t] = \begin{cases} \min\{j + N[i-1, t - jd_i] : 0 \le j \le \lfloor t/d_i \rfloor\} & \text{if } i \ge 2 \\ t & \text{if } i = 1. \text{ OR } t = 0 \end{cases}$$

$N[1 \ldots n, 0 \ldots T]$:

No data dependencies on any other array cells.

***i*-axis (coin type)**

(recall: $N[i,t]$ uses coin types $1..i$)




***t*-axis (target sum remaining)**

# FILLING THE ARRAY

## $N[1 \dots n, 0 \dots T]$:

$$N[i,t] = \begin{cases} \min\{j + N[i-1, t - jd_i] : 0 \le j \le \lfloor t/d_i \rfloor\} & \text{if } i \ge 2 \\ t & \text{if } i = 1. \end{cases}$$

OR $t = 0$

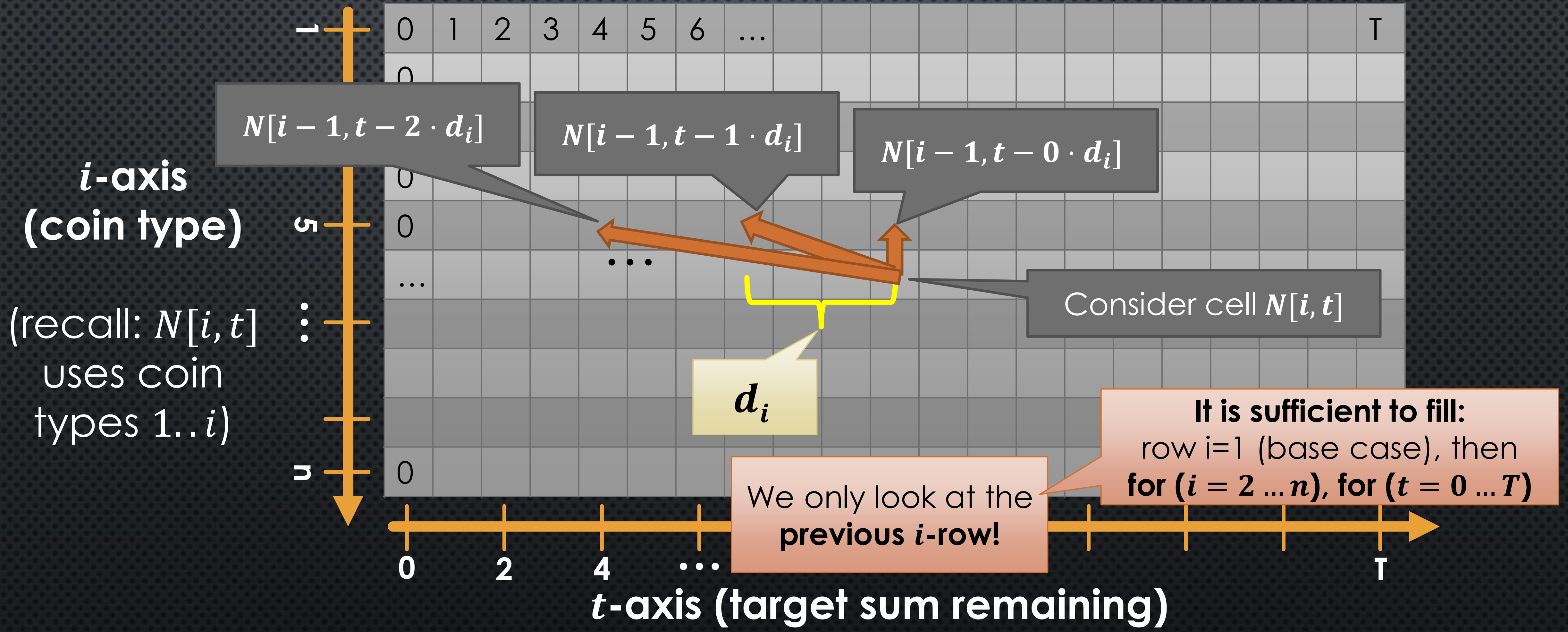
No data dependencies on any other array cells.

***i*-axis (coin type)**

(recall: $N[i,t]$ uses coin types $1..i$)

# FILLING THE ARRAY $N[1 \dots n, 0 \dots T]$:

$$N[i,t] = \begin{cases} \boxed{\min\{j + N[i-1, t - jd_i] : 0 \le j \le \lfloor t/d_i \rfloor\}} & \text{if } i \ge 2 \\ t & \text{if } i = 1. \\ & \text{OR } t = 0 \end{cases}$$



$$N[i,t] = \begin{cases} \min\{j + N[i-1, t-jd_i] : 0 \le j \le \lfloor t/d_i \rfloor\} & \text{if } i \ge 2 \\ t & \text{if } i = 1. \end{cases}$$

```
CoinChangingDP(d[1..n], T)
    N = new table[1..n][0..T]
    J = new table[1..n][0..T]

    for t = 0..T    // base cases where i=1
        N[1][t] = t
        J[1][t] = t

    for i = 2..n    // general cases
        for t = 0..T
            // initially best solution is 0 of d[i]
            N[i][t] = N[i-1][t]
            J[i][t] = 0

            // try j>0 coins of type d[i]
            for j = 1..floor(t / d[i])
                if j + N[i-1][t-j*d[i]] < N[i][t]
                    N[i][t] = j + N[i-1][t-j*d[i]]
                    J[i][t] = j // best is currently j of d[i]

    return N[n][T] // can also return N, J
```

i.e., using coin $d_1 = 1$

$\boldsymbol{J[i,t]}$ = **# of coins** of type $\boldsymbol{d_i}$ used in $N[i,t]$

using *other* coin types

**Compute min{…} over**
$\boldsymbol{j = 0 \ldots \lfloor t/d_i \rfloor}$

# OUTPUTTING **OPTIMAL SET OF COINS**

```
CoinChangingDP_coins(d[1..n], J[1..n][0..T])
    counts = new array[1..n]
    t = T
    for i = n..1
        counts[i] = J[i][t]
        t = t - counts[i]*d[i]

    return counts
```

**Recall $\boldsymbol{J[i,t]}$ = # of coins** of type $\boldsymbol{d_i}$ used in $N[i,t]$

We start at $J[n][T]$ = # of coins of type $d_n$ used in the **optimal solution**

**Exercise for later:**
compute the correct output
**without** using $J[i,t]$
(i.e., using only $N$, $d$, $T$)

```
CoinChangingDP(d[1..n], T)
    N = new table[1..n][0..T]
    J = new table[1..n][0..T]

    for t = 0..T     // base cases where i=1
        N[1][t] = t
        J[1][t] = t

    for i = 2..n     // general cases
        for t = 0..T
            // initially best solution is 0 of d[i]
            N[i][t] = N[i-1][t]
            J[i][t] = 0

            // try j>0 coins of type d[i]
            for j = 1..floor(t / d[i])
                if j + N[i-1][t-j*d[i]] < N[i][t]
                    N[i][t] = j + N[i-1][t-j*d[i]]
                    J[i][t] = j // best is currently j of d[i]

    return N[n][T] // can also return N, J
```

**Time complexity?**

**Unit cost** computational model is reasonable here

Consider instance $I = (d, T)$

Runtime $\boldsymbol{R(I)} \in O\left(\sum_{i=2}^{n} \sum_{t=0}^{T} \left\lfloor \frac{t}{d_i} \right\rfloor\right)$

$$R(I) \in O\left(\sum_{i=2}^{n} \frac{1}{d_i} \sum_{t=0}^{T} t\right)$$

$$R(I) \in O\left(\sum_{i=2}^{n} \frac{1}{d_i} \left(\frac{T(T+1)}{2}\right)\right)$$

$\boldsymbol{R(I) \in O(DT^2)}$

where $D = \sum_{i=2}^{n} \frac{1}{d_i} < n.$

**If T is small, this is much better than brute force**

# POLYNOMIAL TIME

- An algorithm runs in (worst case) **polynomial time** IFF its runtime $R(I)$ on every input is upper bounded by a polynomial in the input size S
- I.e., $R(I) \in O\left(c_0 + c_1 S + c_2 S^2 + c_3 S^3 + \cdots + c_k S^k\right)$
  for **constants $\boldsymbol{k}$ and $\boldsymbol{c_0, \ldots, c_k}$**
- … so is $O(nT^2)$ polynomial in our input size $S$?

# INPUT SIZE

- $S = bits(T) + bits(d_1) + \cdots + bits(d_n)$
- It takes $\lceil \mathbf{log_2}\, \boldsymbol{T} \rceil$ **bits** to store $T$
- It takes $\lceil \mathbf{log_2}\, \boldsymbol{d_i} \rceil$ **bits** to store **each** $d_i$
- **Assume $\boldsymbol{d_i \le T}$ (otherwise $\boldsymbol{d_i}$ cannot be used at all, and should be omitted from the input)**
  - Then we have $\lceil \log_2 d_i \rceil \in O(\log T)$
  - So, $\boldsymbol{S \in O(n \log T)}$

# COMPARING $T(I)$ TO $S$

- Recall $\boldsymbol{R(I) \in O(nT^2)}$ and $\boldsymbol{S \in O(n \log T)}$
- As an example, if $n$ is fixed at 10 and $T$ is allowed to vary, then $\boldsymbol{S \in O(\log T)}$ and $\boldsymbol{R(I) \in O(T^2)}$
  - In this case, $R(I)$ is **exponential in $S$**
- However, if $T$ is fixed at 10 and $n$ is allowed to vary, then $\boldsymbol{S \in O(n)}$ and $\boldsymbol{R(I) \in O(n)}$
  - In this case, $R(I)$ is **linear in $S$**
- So, **large** $n$ and **small** $T$ is where this DP solution shines!

# A BIT MORE ANALYSIS

- Recall $\boldsymbol{R(I) \in O(nT^2)}$ and $\boldsymbol{S \in O(n \log T)}$
- **If** $\boldsymbol{T \in O(n)}$, then $S \in O(n \log n)$ and $R(I) \in O(n^3)$
  - Note $O(n^3)$ is a **smaller** runtime than $O(\boldsymbol{S^3}) = O(n^3 \log n)$
  - And $S^3$ is polynomial in $\boldsymbol{S}$, so $O(n^3)$ is a **polynomial runtime**
- So, **for some inputs** with *relatively small* T, we can get polynomial runtimes!
  - In particular, for $\boldsymbol{T \in O(n^k)}$ where $\boldsymbol{k}$ **is constant,** $R(I) \in O\left(n(n^k)^2\right) = O(n^{2k+1})$ and $S \in O(n \log n^k) = O(n \log n)$
  - And $R(I) \in O(n^{2k+1}) \subseteq O\left((n \log n)^{2k+1}\right) = O(S^{2k+1})$