CS 341: ALGORITHMS

Lecture 9: dynamic programming III

Readings: see website

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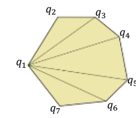
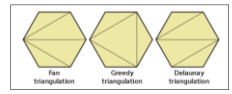
PROBLEM: MINIMUM LENGTH TRIANGULATION

- **Input:** n points $q_1, \ldots, q_n$ in 2D space that form a **convex** n-gon P
  - Assume points are **sorted clockwise** around the center of P

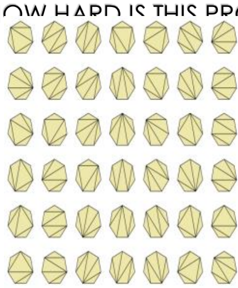


- **Find:** a triangulation of P such that the sum of the perimeters of the $n-2$ triangles is minimized



- **Output:** the **sum** of the **perimeters** of the triangles in P

HOW HARD IS THIS PROBLEM?



How many triangulations are there?

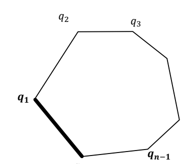
Number of triangulations of a convex n-gon = the $(n-2)$**nd Catalan number**

This is $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$

It can be shown that $C_{n-2} \in \Theta(4^n/(n-2)^{3/2})$

PROBLEM DECOMPOSITION

The edge q_nq_1 is in a triangle with a third vertex q_k, where $k \in \{2, \ldots, n-1\}$.



PROBLEM DECOMPOSITION

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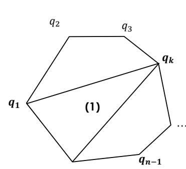
For a given k, we have:

the triangle $q_1q_kq_n$, (1)



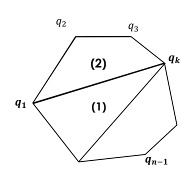
PROBLEM DECOMPOSITION

The edge q_nq_1 is in a triangle with a third vertex q_k, where $k \in \{2, \ldots, n-1\}$.

For a given k, we have:

the triangle $q_1q_kq_n$, (1)

the polygon with vertices $q_1, \ldots, q_k$. (2)

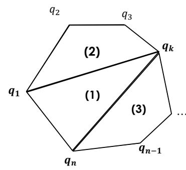


PROBLEM DECOMPOSITION

The edge $q_n q_1$ is in a triangle with a third vertex q_k, where $k \in \{2, \ldots, n-1\}$.

For a given k, we have:

- the triangle $q_1 q_k q_n$, (1)
- the polygon with vertices $q_1, \ldots, q_k$, (2)
- the polygon with vertices $q_k, \ldots, q_n$. (3)



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- the triangle $q_1 q_k q_n$, (1)
- the polygon with vertices $q_1, \ldots, q_k$, (2)
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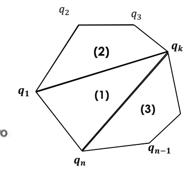
The optimal solution will consist of optimal solutions to the two subproblems in (2) and (3), along with the triangle in (1).



PROBLEM DECOMPOSITION

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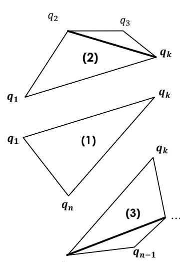
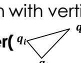
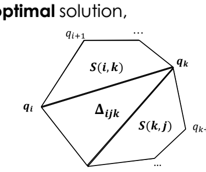


RECURRENCE RELATION

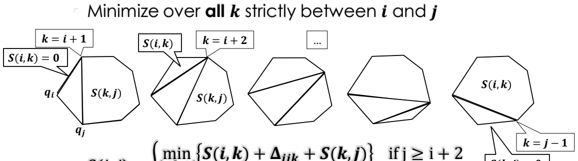
- Let $S(i,j)$ = optimal solution to the subproblem consisting of the polygon with vertices $q_i \ldots q_j$
- Let Δ_{ijk} denote **perimeter(**  **)**

- **If** a given **triangle q_i,q_j,q_k** is in the **optimal** solution, then $S(i,j) = S(i,k) + \Delta_{ijk} + S(k,j)$



RECURRENCE RELATION

- But we don't know the optimal k
  - Minimize over **all k** strictly between i and j

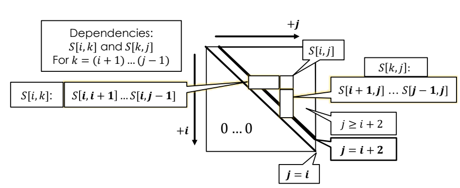


$$S(i,j) = \begin{cases} \min\limits_{i<k<j} \{S(i,k) + \Delta_{ijk} + S(k,j)\} & \text{if } j \geq i+2 \\ 0 & \text{otherwise} \end{cases}$$

FILLING IN THE TABLE

$$S(i,j) = \begin{cases} \min\limits_{i<k<j} \{S(i,k) + \Delta_{ijk} + S(k,j)\} & \text{if } j \geq i+2 \\ 0 & \text{otherwise} \end{cases}$$

- Table $S[1..n, 1..n]$ of solutions to $S(i,j)$ for all $i, j \in \{1..n\}$



We depend on **larger i** And **same i but smaller j**	**What's a correct fill order?** **for $i = n..1$, for $j = 1..n$**

RUNTIME
WORD RAM MODEL

$$S(i,j) = \begin{cases} \min_{i<k<j} \{\boldsymbol{S(i,k)} + \boldsymbol{\Delta_{ijk}} + \boldsymbol{S(k,j)}\} & \text{if } j \geq i+2 \\ 0 & \text{otherwise} \end{cases}$$

- Number of subproblems: n^2
- Time to solve subproblem $S(i,j)$: $O(j-i) \subseteq O(n)$
- So total runtime is in $O(n^3)$
  - Some effort needed to show $\Omega(n^3)$, since so many subproblems are base cases, which take $\Theta(1)$ steps
- **Incidentally**, this is polynomial time (in the input size)
  - But basic runtime analysis does **not** require such an argument

PROBLEM: LONGEST COMMON SUBSEQUENCE (LCS)

Problem 5.3
Longest Common Subsequence
Instance: *Two sequences $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_n)$ over some finite alphabet Γ.*
Find: *A maximum length sequence Z that is a subsequence of both X and Y.*

$Z = (z_1, \ldots, z_\ell)$ is a **subsequence** of X if there exist indices $1 \leq i_1 < \cdots < i_\ell \leq m$ such that $z_j = x_{i_j}$, $1 \leq j \leq \ell$.
Similarly, Z is a subsequence of Y if there exist (possibly different) indices $1 \leq h_1 < \cdots < h_\ell \leq n$ such that $z_j = y_{h_j}$, $1 \leq j \leq \ell$.

Let's **first** solve for the **length** of the LCS

EXAMPLES

- X=aaaaa Y=bbbbb Z=LCS(X,Y)=?
  - Z=ϵ (empty sequence)
- X=abcde Y=bcd Z=LCS(X,Y)=?
  - Z=bcd
- X=abcde Y=labef Z=LCS(X,Y)=?
  - Z=abe

POSSIBLE GREEDY SOLUTIONS?

- Alg: for each $x_i \in X$, try to choose a **matching** $y_j \in Y$ that is **to the right** of all previously chosen y_j values
  - X=**a**bcde Y=l**a**bef
  - X=**ab**cde Y=l**ab**ef
  - X=**ab**cde Y=l**ab**ef [no suitable y_j found]
  - X=**abc**de Y=l**ab**ef [no suitable y_j found]
  - X=**abcde** Y=l**abe**f
  - Z=abe Optimal?

POSSIBLE GREEDY SOLUTIONS?

- Alg: for each $x_i \in X$, try to choose a **matching** $y_j \in Y$ that is **to the right** of all previously chosen y_j values
  - X=**a**zbracadabra Y=**a**bracadabraz
  - X=**az**bracadabra Y=**a**bracadabra**z**
  - X= **z**bracadabra Y=**a**bracadabra**z** [no y_j after **z**]
  - racadabra Y=**a**bracadabra**z** [no y_j after **z**]
  - Z=a Optimal?

Blindly taking z is bad. **How to decide** whether to take or leave z?

Try **both** possibilities! (Brute force / dynamic programming)

Similar greedy alg that goes right-to-left works for this input, but fails for other inputs.

DEFINING SUBPROBLEMS

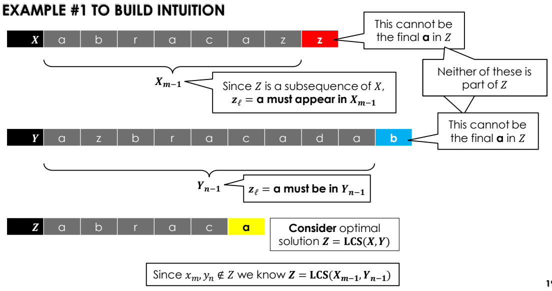
- **Full problem:** $|\mathbf{LCS}(\boldsymbol{X},\boldsymbol{Y})|$ (i.e., **length** of LCS)
  - Reduce size by taking **prefixes** of X or Y
  - Let $\boldsymbol{X_i} = (\boldsymbol{x_1}, \ldots, \boldsymbol{x_i})$ and $\boldsymbol{Y_i} = (\boldsymbol{y_1}, \ldots, \boldsymbol{y_i})$

X_m	x_1	x_2	x_3	x_4	…				x_{m-1}	x_m
X_4	x_1	x_2	x_3	x_4						

  Note $X = X_m$ and $Y = Y_n$
- **Subproblem:** $|\mathbf{LCS}(\boldsymbol{X_i},\boldsymbol{Y_j})|$
- **Shrinking the problem:** remove the **last letter** of X or Y

BUILDING SOLUTIONS FROM SUBPROBLEMS

EXAMPLE #1 TO BUILD INTUITION



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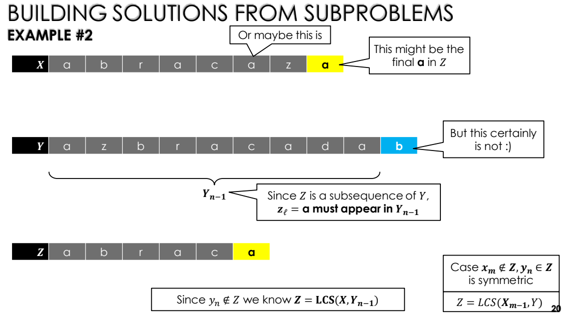
BUILDING SOLUTIONS FROM SUBPROBLEMS

EXAMPLE #2



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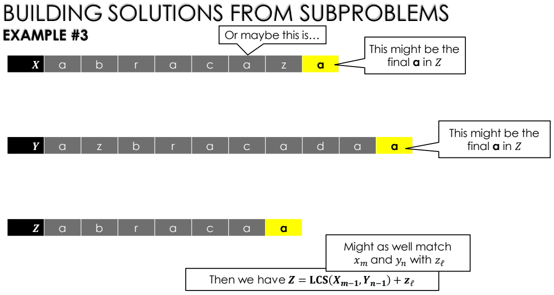
BUILDING SOLUTIONS FROM SUBPROBLEMS

EXAMPLE #3



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SUMMARIZING CASES

- z_ℓ matches **neither** x_m nor y_n $\quad Z = \mathrm{LCS}(X_{m-1}, Y_{n-1})$
- z_ℓ matches x_m but not y_n $\quad Z = \mathrm{LCS}(X_m, Y_{n-1})$
- z_ℓ matches y_n but not x_m $\quad Z = \mathrm{LCS}(X_{m-1}, Y_n)$
- z_ℓ matches **both** $\quad Z = \mathrm{LCS}(X_{m-1}, Y_{n-1}) + \boldsymbol{z_\ell}$
- **… but we don't know z_ℓ**
  - Try all cases and maximize
  - **Careful: last case is only valid if $x_m = y_n$**
- Also note $x_m = y_n$ **only holds in the last case**
  - Cases 2&3: trivial
  - Case 1: if $x_m = y_n \neq z_\ell$ then we can improve Z (contra)

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DERIVING A RECURRENCE

Recall $Z = \mathrm{LCS}(X_m, Y_n)$

- z_ℓ matches **neither** x_m nor y_n $\quad (x_m \neq y_n) \quad Z = \mathrm{LCS}(X_{m-1}, Y_{n-1})$
- z_ℓ matches x_m but not y_n $\quad (x_m \neq y_n) \quad Z = \mathrm{LCS}(X_m, Y_{n-1})$
- z_ℓ matches y_n but not x_m $\quad (x_m \neq y_n) \quad Z = \mathrm{LCS}(X_{m-1}, Y_n)$
- z_ℓ matches **both** $\quad (x_m = y_n) \quad Z = \mathrm{LCS}(X_{m-1}, Y_{n-1}) + z_\ell$
- **Let $c(i,j) = |LCS(X_i, Y_j)|$**
- Brainstorming sensible base cases
  - $i = 0$ one string is empty, so $c(0, j) = 0$ (similarly for $j = 0$)
- General cases

$c(i,j) = c(i-1, j-1) + 1$	if $x_m = y_n$
$c(i,j) = \max\{c(i-1, j-1), c(i, j-1), c(i-1, j)\}$	if $x_m \neq y_n$

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RECURRENCE

- Combining expressions

$$c(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c(i-1, j-1) + 1 & \text{if } i, j \geq 1 \text{ and } x_i = y_j \\ \max\{c(i, j-1), c(i-1, j), c(i-1, j-1)\} & \text{if } i, j \geq 1 \text{ and } x_i \neq y_j \end{cases}$$

- Can simplify!
  - Observe $c(i-1, j-1) \leq c(i-1, j)$ (former is a subproblem of the latter)

$$c(i,j) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c(i-1, j-1) + 1 & \text{if } i, j \geq 1 \text{ and } x_i = y_j \\ \boldsymbol{\max\{c(i, j-1), c(i-1, j)\}} & \text{if } i, j \geq 1 \text{ and } x_i \neq y_j \end{cases}$$

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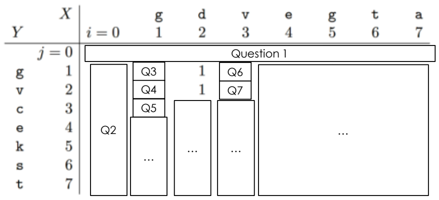
Suppose X = **gdvegta** and Y = **gvcekst**

$$c(i,j)=\begin{cases}0 & \text{if } i=0 \text{ or } j=0\\ c(i-1,j-1)+1 & \text{if } i,j\ge 1 \text{ and } x_i=y_j\\ \max\{c(i,j-1),c(i-1,j)\} & \text{if } i,j\ge 1 \text{ and } x_i\ne y_j\end{cases}$$



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Y \ X			g	d	v	e	g	t	a
		$i=0$	1	2	3	4	5	6	7
	$j=0$	0	0	0	0	0	0	0	0
g	1	0	1	1	1	1	1	1	1
v	2	0	1	1	2	2	2	2	2
c	3	0	1	1	2	2	2	2	2
e	4	0	1	1	2	3	3	3	3
k	5	0	1	1	2	3	3	3	3
s	6	0	1	1	2	3	3	3	3
t	7	0	1	1	2	3	3	4	4

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PSEUDOCODE

$$c(i,j)=\begin{cases}0 & \text{if } i=0 \text{ or } j=0\\ c(i-1,j-1)+1 & \text{if } i,j\ge 1 \text{ and } x_i=y_j\\ \max\{c(i,j-1),c(i-1,j)\} & \text{if } i,j\ge 1 \text{ and } x_i\ne y_j\end{cases}$$

Algorithm: $LCS1(X=(x_1,\ldots,x_m), Y=(y_1,\ldots,y_n))$

```
for i ← 0 to m
    c[i,0] ← 0
for j ← 0 to n
    c[0,j] ← 0
for i ← 1 to m
    for j ← 1 to n
        if x_i = y_j
        then c[i,j] ← c[i-1,j-1] + 1
        else c[i,j] ← max{c[i,j-1], c[i-1,j]}
return (c[m,n]);
```

Complexity: Space? Time? (word RAM model)
$\Theta(nm)$ for both

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COMPUTING THE **LCS**

NOT JUST ITS LENGTH

To make it easy to find the actual LCS (not just its length),

Consider **which table entry** was used to calculate $c[i,j]$

$$=\begin{cases}0 & \text{if } i=0 \text{ or } j=0\\ c(i-1,j-1)+1 & \text{if } i,j\ge 1 \text{ and } x_i=y_j\\ \max\{c(i,j-1),c(i-1,j)\} & \text{if } i,j\ge 1 \text{ and } x_i\ne y_j\end{cases}$$

We store the **direction** to that entry in an array $\pi[i,j]$

Case 1: $c(i,j)=c(i,j-1)$	**Case 2:** $c(i,j)=c(i-1,j)$	**Case 3:** $c(i,j)=c(i-1,j-1)+1$
We store "J" in $\pi[i,j]$ to indicate **decrementing** j (to get $i,j-1$)	We store "I" in $\pi[i,j]$ to indicate decrementing i (to get $i-1,j$)	We store "IJ" in $\pi[i,j]$ to indicate decrementing **both** i and j
		Recall in this case, $x_i=y_j$ so we **include x_i in the LCS**

In our example table we just **draw an arrow** to the entry…

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SAVING THE DIRECTION TO THE **PREDECESSOR** SUBPROBLEM π

```
LCS2(X[1..m], Y[1..n])
    c = new array[0..m][0..n]
    π = new array[0..m][0..n]

    for i = 0..m do c[i][0] = 0
    for j = 0..n do c[0][j] = 0

    for i = 1..m
        for j = 1..n
            if X[i] = Y[j]
                c[i][j] = c[i-1][j-1] + 1
                π[i][j] = "IJ"
            else if c[i][j-1] > c[i-1][j]
                c[i][j] = c[i][j-1]
                π[i][j] = "J"
            else // c[i][j-1] <= c[i-1][j]
                c[i][j] = c[i-1][j]
                π[i][j] = "I"

    return c, π
```

Case: $c(i,j)=c(i-1,j-1)+1$
We store "IJ" in $\pi[i,j]$ to indicate decrementing **both** i and j
Recall in this case, $x_i=y_j$ so we **include x_i in the LCS**

Case: $c(i,j)=c(i,j-1)$
We store "J" in $\pi[i,j]$ to indicate **decrementing** j (to get $i,j-1$)

Case: $c(i,j)=c(i-1,j)$
We store "I" in $\pi[i,j]$ to indicate decrementing i (to get $i-1,j$)

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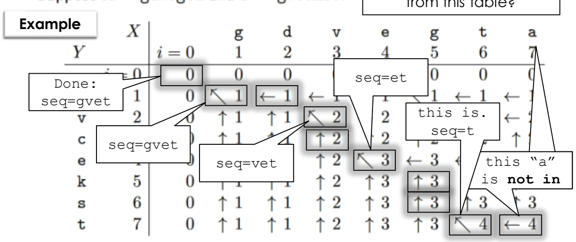
Suppose X = gdvegta and Y = gvcekst.

How to obtain LCS=gvet from this table?

Example



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FOLLOWING PREDECESSORS TO COMPUTE THE **LCS**

```
FindLCS(c[0..m][0..n], π[0..m][0..n], X[0..m])
    lcs = new string
    i = m
    j = n

    while i>0 and j>0
        if π[i][j] == "IJ"
            lcs.append(X[i])
            i--
            j--
        else if π[i][j] == "J"
            j--
        else // π[i][j] == "I"
            i--

    return reverse(lcs)
```

Complexities of this trace-back algo: Space? Time? (word RAM model)

space: O(n+m) words

time: O(n+m)

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UNLIKELY TO GET THIS FAR

So this is likely just an exercise for you…

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COIN CHANGING

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Coin Changing

Problem 5.2

Coin Changing

Instance: *A list of* **coin denominations**, $1 = d_1, d_2, \ldots, d_n$, *and a positive integer* T, *which is called the* **target sum**.

Find: *An* n*-tuple of non-negative integers, say* $A = [a_1, \ldots, a_n]$, *such that* $T = \sum_{i=1}^{n} a_i d_i$ *and such that* $N = \sum_{i=1}^{n} a_i$ *is minimized.*

There **is** a denomination with **unit value**!

What subproblems should be considered?

What table of values should we fill in?

In 0-1 knapsack, we only considered **two subproblems** in our recurrence: **taking an item, or not.**

Here we can do **more than** *use a coin denomination or not*.

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Let $N[i, t]$ denote the optimal solution to the subproblem consisting of the first i coin denominations $d_1, \ldots, d_i$ and target sum t.

Exploring: some sensible base case(s)?

General case:
What are the different ways we could use coin denomination $\boldsymbol{d_i}$?
What subproblems / solutions should we use?

Final recurrence relation

35

Let $N[i, t]$ denote the optimal solution to the subproblem consisting of the first i coin denominations $d_1, \ldots, d_i$ and target sum t.

Since $d_1 = 1$, we immediately have $N[1, t] = t$ for all t.

Also $\boldsymbol{N[i, 0] = 0}$ for all i

General case:
What are the different ways we could use coin denomination $\boldsymbol{d_i}$?
What subproblems / solutions should we use?

Final recurrence relation

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Let $N[i,t]$ denote the optimal solution to the subproblem consisting of the first i coin denominations $d_1, \ldots, d_i$ and target sum t.

Also $N[i,0] = 0$ for all i

Since $d_1 = 1$, we immediately have $N[1,t] = t$ for all t.

For $i \geq 2$, the number of coins of denomination d_i is an integer j where $0 \leq j \leq \lfloor t/d_i \rfloor$.

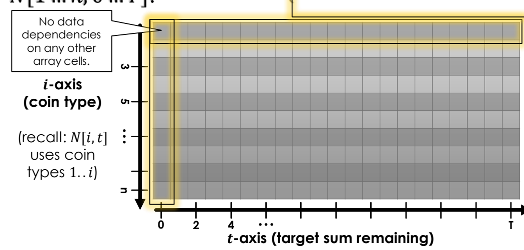
If we use j coins of denomination d_i, then the target sum is reduced to $t - jd_i$, which we must achieve using the first $i-1$ coin denominations.

Thus we have the following recurrence relation:

$$N[i,t] = \begin{cases} \min\{j + N[i-1, t-jd_i] : 0 \leq j \leq \lfloor t/d_i \rfloor\} & \text{if } i \geq 2 \\ t & \text{if } i = 1 \text{ OR } t = 0 \end{cases}$$

FILLING THE ARRAY $N[1 \ldots n, 0 \ldots T]$:

$$N[i,t] = \begin{cases} \min\{j + N[i-1, t-jd_i] : 0 \leq j \leq \lfloor t/d_i \rfloor\} & \text{if } i \geq 2 \\ t & \text{if } i = 1. \text{ OR } t = 0 \end{cases}$$



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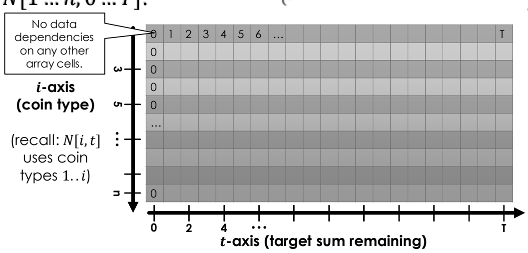

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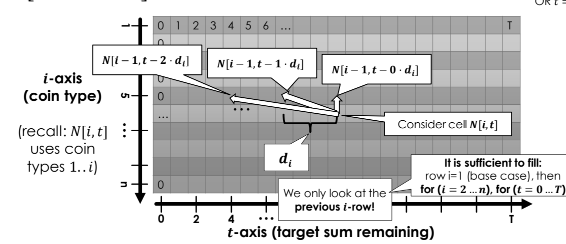

$$N[i,t] = \begin{cases} \min\{j + N[i-1, t-jd_i] : 0 \leq j \leq \lfloor t/d_i \rfloor\} & \text{if } i \geq 2 \\ t & \text{if } i = 1. \end{cases}$$

```
CoinChangingDP(d[1..n], T)
    N = new table[1..n][0..T]
    J = new table[1..n][0..T]

    for t = 0..T    // base cases where i=1
        N[1][t] = t
        J[1][t] = t

    for i = 2..n    // general cases
        for t = 0..T
            // initially best solution is 0 of d[i]
            N[i][t] = N[i-1][t]
            J[i][t] = 0

            // try j>0 coins of type d[i]
            for j = 1..floor(t / d[i])
                if j + N[i-1][t-j*d[i]] < N[i][t]
                    N[i][t] = j + N[i-1][t-j*d[i]]
                    J[i][t] = j // best is currently j of d[i]

    return N[n][T] // can also return N, J
```

i.e., using coin $d_1 = 1$

$J[i,t]$ = **# of coins** of type d_i used in $N[i,t]$

using *other* coin types

Compute min{…} over $j = 0 \ldots \lfloor t/d_i \rfloor$

OUTPUTTING **OPTIMAL SET OF COINS**

```
CoinChangingDP_coins(d[1..n], J[1..n][0..T])
    counts = new array[1..n]
    t = T
    for i = n..1
        counts[i] = J[i][t]
        t = t - counts[i]*d[i]

    return counts
```

Recall $J[i,t]$ = # of coins of type d_i used in $N[i,t]$

We start at $J[n][T]$ = # of coins of type d_n used in the **optimal solution**

Exercise for later:
compute the correct output
without using $J[i,t]$
(i.e., using only N, d, T)

```
CoinChangingDP(d[1..n], T)
    N = new table[1..n][0..T]
    J = new table[1..n][0..T]

    for t = 0..T    // base cases where i=1
        N[1][t] = t
        J[1][t] = t

    for i = 2..n    // general cases
        for t = 0..T
            // initially best solution is 0 of d[i]
            N[i][t] = N[i-1][t]
            J[i][t] = 0

            // try j>0 coins of type d[i]
            for j = 1..floor(t / d[i])
                if j + N[i-1][t-j*d[i]] < N[i][t]
                    N[i][t] = j + N[i-1][t-j*d[i]]
                    J[i][t] = j // best is currently j of d[i]

    return N[n][T] // can also return N, J
```

Time complexity?

Unit cost computational model is reasonable here

Consider instance $I = (d, T)$

Runtime $\boldsymbol{R(I)} \in O\left(\sum_{i=2}^{n}\sum_{t=0}^{T}\left\lfloor\frac{t}{d_i}\right\rfloor\right)$

$$R(I) \in O\left(\sum_{i=2}^{n}\frac{1}{d_i}\sum_{t=0}^{T} t\right)$$

$$R(I) \in O\left(\sum_{i=2}^{n}\frac{1}{d_i}\left(\frac{T(T+1)}{2}\right)\right)$$

$\boldsymbol{R(I) \in O(DT^2)}$
where $D = \sum_{i=2}^{n}\frac{1}{d_i} < n.$

If T is small, this is much better than brute force

MEMOIZATION: AN ALTERNATIVE TO DP

Recall that the goal of dynamic programming is to eliminate solving subproblems more than once.

Memoization is another way to accomplish the same goal.

Memoization is a recursive algorithm based on same recurrence relation as would be used by a dynamic programming algorithm.

The idea is to remember which subproblems have been solved; if the same subproblem is encountered more than once during the recursion, the solution will be looked up in a table rather than being re-calculated.

This is easy to do if initialize a table of all possible subproblems having the value undefined in every entry.

Whenever a subproblem is solved, the table entry is updated.

EXAMPLE: USING MEMOIZATION TO COMPUTE FIBONACCI NUMBERS EFFICIENTLY

main
for $i \leftarrow 2$ **to** n
  do $M[i] \leftarrow -1$
return $(\textit{RecFib}(n))$

procedure $\textit{RecFib}(n)$
 if $n = 0$ **then** $f \leftarrow 0$
  else if $n = 1$ **then** $f \leftarrow 1$
  else if $M[n] \neq -1$ **then** $f \leftarrow M[n]$
  else $\begin{cases} f_1 \leftarrow \textit{RecFib}(n-1) \\ f_2 \leftarrow \textit{RecFib}(n-2) \\ f \leftarrow f_1 + f_2 \\ M[n] \leftarrow f \end{cases}$
 return (f);

If **M[n]** is already computed, **don't recurse!**

VISUALIZING MEMOIZATION

If **M[n]** is already computed, **don't recurse!**

procedure $\textit{RecFib}(n)$
 if $n = 0$ **then** $f \leftarrow 0$
  else if $n = 1$ **then** $f \leftarrow 1$
  else if $M[n] \neq -1$ **then** $f \leftarrow M[n]$
  else $\begin{cases} f_1 \leftarrow \textit{RecFib}(n-1) \\ f_2 \leftarrow \textit{RecFib}(n-2) \\ f \leftarrow f_1 + f_2 \\ M[n] \leftarrow f \end{cases}$
 return (f);