Lecture 12: Graph Algorithms III

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Overview

- Directed Graphs
  - Reachability
  - BFS/DFS trees
  - Directed Acyclic Graphs (DAGs) & Topological Sort
  - Strongly Connected Components
- Acknowledgements

Directed Graphs

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- Useful to model situations with asymetry:
  - web page links
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  - dependencies in parallel computation
- Notation:
  - $\deg_{in}(u) = \#$ vertices $s \in V$ such that $(s, u) \in E$ (in-degree/fanin)
  - $\deg_{out}(u) = \#$ vertices $t \in V$ such that $(u, t) \in E$ (out-degree/fanout)

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- Just as with undirected graphs, we will find $O(n + m)$ time algorithms for these and other problems.

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- **Input:** directed graph $G(V, E)$, $s \in V$
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- Could use either **BFS** or **DFS** for this question. We will use DFS.

Checking Reachability

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- EXPLORE(u, visited, p, S, F, τ):
  1. $S[u] = \tau$, and $\tau \leftarrow \tau + 1$
  2. for each $v \in N_{out}(u)$: (only outgoing neighbors)
     - If visited$[v] = 0$, then
       visited$[v] = 1$, $p[v] = u$
       and EXPLORE(v, visited, p, S, F, τ).
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- Main algorithm:
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  3. EXPLORE(s, visited, p, S, F, τ)
- Time complexity $O(n + m)$, and similarly to undirected case, t reachable iff visited$[t] = 1$.

Directed Cuts

- Set of all visited vertices forms a “directed cut”
  - no outgoing edges
  - possibly incoming edges

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- Still plenty of structure left:

  Parenthesis lemma still holds!

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Shortest paths from source.

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- Very useful to find ordering of vertices so that all edges "go forward"

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- ($\Leftarrow$) given topological ordering, no edge goes backwards, therefore no cycles

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- **Proof of indegree zero vertex:**
  - Suppose (for sake of contradiction) that every vertex u has $\deg_{in}(u) \geq 1$.
  - Starting from vertex $t =: u_0$, go to an in-neighbour u_1, and then to an in-neighbour u_2 and so on. (possible since $\deg_{in}(u_i) > 0$)
  - Since graph is finite, at some point must repeat a vertex $\Rightarrow$ found a cycle. (contradiction)

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- Can use above procedure to topologically sort a DAG (exercise)

Constructing a Topological Ordering

- Algorithm:
  1. Run DFS on the whole graph
  2. Output the ordering with decreasing finishing time
  3. Check if this is a topological ordering. If not, return **not acyclic**.

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- We have 2 cases:
  - Case 1: $S[v] < S[u]$.
  - Since graph is a DAG (no cycles) u not reachable from v
  - Hence u not descendant of v. By parenthesis property, must have

$$[S[v], F[v]] \cap [S[u], F[u]] = \emptyset \Rightarrow F[v] < F[u]$$

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- We have 2 cases:
  - Case 2: $S[v] > S[u]$.
  - Since visited$[v] = 0$ when we start u and $(u, v) \in E$, v will be a descendant of u in DFS tree.
  - Parenthesis lemma implies $[S[v], F[v]] \subset [S[u], F[u]]$

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- Running time: $O(n + m)$ (can obtain sorted list within algorithm)

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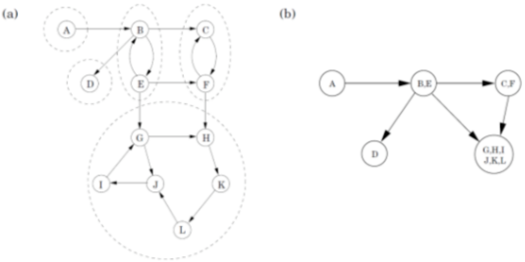

picture from
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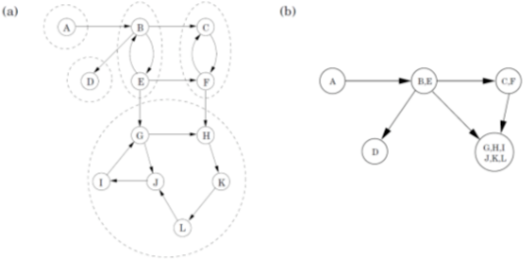

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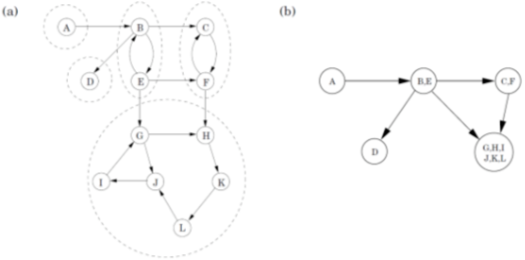

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Exercise: Prove this

- **Observation 2:** general directed graph is a DAG on its SCCs!
- Can we find a "topological sorting" of the SCCs? Need to find one component...

Strongly Connected Components

- **Idea 1:** If we started a DFS/BFS in a "sink component" Γ (with no outgoing edges), then we will certainly find only Γ and then we can recurse on $G \setminus \Gamma$.

Can we find such a component?

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  Doesn't work: node of earliest finishing time need not be in sink component.

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- **Observation 3:** note that node with largest finishing time will be in a *source component*!

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  - Case 1: first visited vertex $u \in \Gamma \sqcup \Gamma'$ is in Γ
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    - Since vertices in $\Gamma \sqcup \Gamma'$ are reachable from u, all vertices in $\Gamma \sqcup \Gamma'$ will be finished before u, so largest finishing time will be of u
  - Case 2: first visited vertex $u \in \Gamma \sqcup \Gamma'$ is in Γ'
    - Since vertices from Γ unreachable from Γ', DFS needs to finish exploring Γ' before starting any vertex in Γ.

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- Can follow the ordering of the finishing times of DFS applied to G^R to get our sink components in G! (or vice-versa!)

Strongly Connected Components - Algorithm

- **Input:** directed graph $G(V, E)$
- **Output:** Strongly connected components of G
- **Algorithm:**
  1. Run DFS on G using arbitrary ordering of vertices
  2. Order vertices by decreasing order of finishing times, label vertices by $u_1, \ldots, u_n$ with $F[u_i] > F[u_{i+1}]$
  3. Reverse G to obtain G^R
  4. Follow ordering in Step 2 to explore G^R and cut out one SCC at a time
     - Let $\gamma = 1$ (counts # SCCs)
     - For $1 \leq i \leq n$ do:
       If visited$[u_i] = 0$, then: $DFS(G^R, u_i)$ and mark all vertices reachable from u_i in G^R to be in component Γ_γ. Then set $\gamma \leftarrow \gamma + 1$

Acknowledgement

- Based on Prof. Lau's lecture 07

  `https://cs.uwaterloo.ca/~lapchi/cs341/notes/L07.pdf`

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