Lecture 16: Max-Flow & Min-Cut

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November 7, 2023

Overview

- Paths, Flows & Cuts
  - Paths
  - Flows
  - Cuts
- Ford-Fulkerson Algorithm
  - Residual Graph
  - Main Algorithm
- Acknowledgements

Paths: Measure of "Resilience"

- Given (directed) graph $G(V, E)$, we would like to know how "resilient" it may be
  - Is G (strongly) connected?
  - How many edges does one need to remove to disconnect it?
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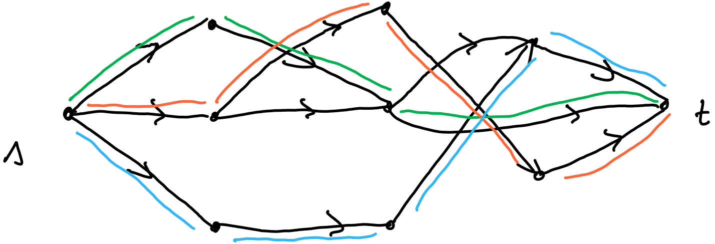

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Does it have 2 *edge-disjoint* $s \to t$ paths?

How many edge-disjoint paths does it have?

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- More generally, can consider weighted directed graphs
  - networks: edge weights are how much data can go through
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Weighted version is (almost) the *maximum flow* problem.

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  Weighted version is (almost) the *maximum flow* problem.
- How to generalize notion of edge-disjoint in weighted version?

Flows

- We will now think of a weighted graph $G(V, E, c)$, where $c : E \to \mathbb{R}_{>0}$ (the weight function) is giving the *capacity* of an edge

  If we have $c : E \to \mathbb{N}$ then
  - Think of capacity as number of lanes in a street/highway
  - Or think of $G(V, E, c)$ as unweighted graph with $c((u, v))$ being the number of distinct $u \to v$ edges

Flows

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- An $s \to t$ flow is a function $f : E \to \mathbb{R}_{\geq 0}$ with the properties:

  1. *Capacity constraints*: $0 \leq f(e) \leq c(e)$ for all $e \in E$
  2. *Flow conservation*: $f_{\text{in}}(u) = f_{\text{out}}(u)$ for each $u \in V \setminus \{s, t\}$, where

$$f_{\text{in}}(u) := \sum_{w \in N_{in}(u)} f(w, u), \quad \text{and} \quad f_{\text{out}}(u) := \sum_{w \in N_{out}(u)} f(u, w)$$

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- *Max-flow* problem:
  - **Input:** directed graph $G(V, E, c)$, with $c : E \to \mathbb{R}_{>0}$, vertices $s, t \in V$
  - **Output:** an $s \to t$ flow with maximum value.

Example (from Jeff Erickson's book)

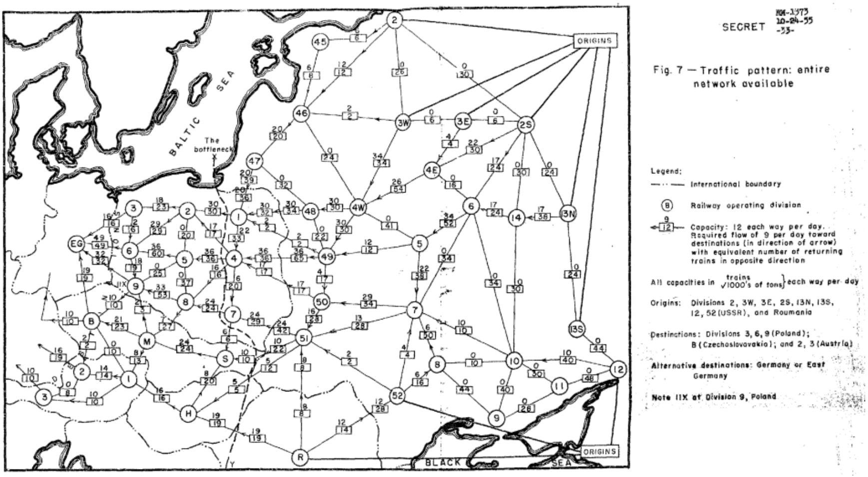

Figure 10.1. Harris and Ross's map of the Warsaw Pact rail network. (See Image Credits at the end of the book.)

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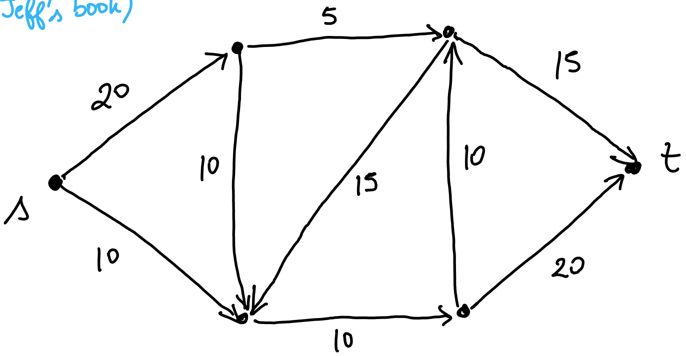
Integral flow f (i.e. $f : E \to \mathbb{N}$) with $\mathrm{value}(f) = k$ corresponds to k edge-disjoint paths in the unweighted graph $G(V, E, c)$ above

- Think of edge e with $f(e) = h$ as the collections of paths using h lanes in highway
- flow conservation $\leftrightarrow$ # cars entering vertex u = # cars leaving vertex u
- capacity constraints $\leftrightarrow$ each car gets one lane in highway

Example

(also in Jeff's book)



Path decomposition lemma

Lemma (Path Decomposition Lemma)

Let G be a weighted DAG with integral weights. Let f be an integral $s \to t$ flow, with $f_{\text{in}}(s) = 0$ and $\text{value}(f) = k$. Then, there are $s \to t$ paths $P_1, \ldots, P_k$ such that each edge e appears in $f(e)$ of these paths.

Remark: for full "flow decomposition theorem" see Jeff Erickson's book, chapter 10.

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- an $s - t$ cut is a cut $(S, V \setminus S)$ such that $s \in S$ and $t \notin S$.
  - Capacity of cut:

$$C_{\mathrm{out}}(S) := \sum_{e \in \delta_{out}(S)} c(e)$$

where $\delta_{out}(S) = \{(u, v) \in E \mid u \in S, v \notin S\}$

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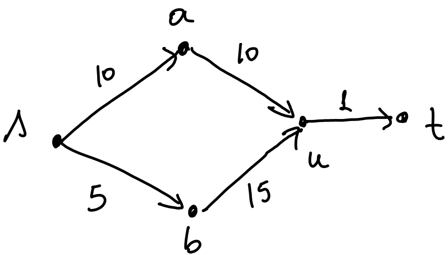
  where $\delta_{out}(S) = \{(u, v) \in E \mid u \in S, v \notin S\}$
  - By path decomposition lemma or flow conservation, can prove that
  $$\mathrm{value}(f) \leq C_{\mathrm{out}}(S)$$
  for any flow f and cut S.

Example



note that s-t cut $\{s, a, b, u\}$ gives better upper bound than $\{s\}$.

Max-Flow Min-Cut Theorem

- Capacity of cuts are an upper bound for flows.

Is this a tight upper bound?

Theorem (Max-Flow Min-Cut Theorem)

The value of the maximum $s - t$ flow equals the minimum capacity among all cuts.

$$\max_{f \text{ } s-t \text{ flow}} \mathrm{value}(f) = \min_{S \text{ is } s-t \text{ cut}} C_{\mathrm{out}}(S)$$

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Theorem (Max-Flow Min-Cut Theorem)

The value of the maximum $s-t$ flow equals the minimum capacity among all cuts.

$$\max_{f \ s-t \ flow} \mathrm{value}(f) = \min_{S \ is \ s-t \ cut} C_{\mathrm{out}}(S)$$

- We will give an algorithmic proof of this theorem, that solves the max-flow and the min-cut problem at the same time.

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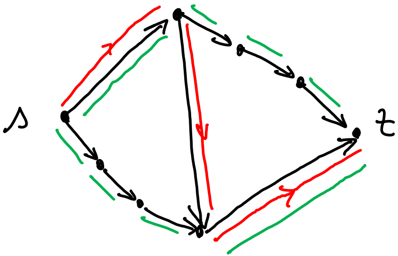
  Main idea behind Ford-Fulkerson.

- Augment the flow by finding "*augmenting path*" which *increases* **total amount of flow**

Example

(bad greedy example)



all capacities 1.

Shortest path in red would block max flow in green

Residual Graph

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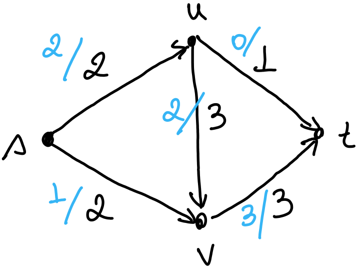
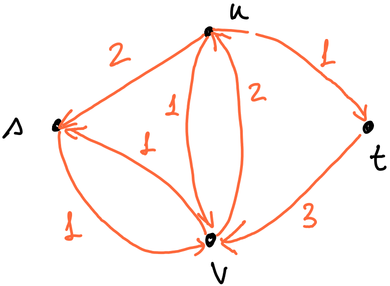
- Given $G(V, E, c)$ and $s \rightarrow t$ flow f on G, define the *residual graph* G_f as follows:
    - $V(G_f) = V(G)$
    - For each $(u, v) =: e \in E$ add edges
        - (u, v) to G_f with capacity $c(e) - f(e)$ (forward edges)
        - (v, u) to G_f with capacity $f(e)$ (backward edges)

Example

Graph & flow f



Residual graph



Augmenting Path

- An *augmenting path* with respect to a flow f is simply an $s \to t$ path[1] in G_f

[1]By path here we mean a simple path, and not a walk.

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- An *augmenting path* with respect to a flow f is simply an $s \to t$ path[1] in G_f
- Given augmenting path P in G_f, want to push *as much flow as possible* through it:

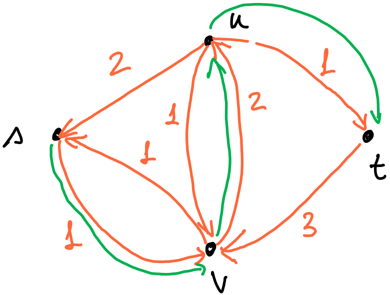
$$\text{bottleneck}(P, f) := \text{minimum capacity of edge of } P \text{ in } G_f$$

[1] By path here we mean a simple path, and not a walk.

Example

In previous residual graph, have following augmenting path

$\text{bottleneck}(P, f) = 1$



Improving the Flow

- **Input:** flow f and an augmenting path P in G_f
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augment(f, P) :

- Let $b :=$ bottleneck(P, f) and $f'(e) = f(e)$ for all $e \in E$
- for each $e := (u, v) \in P$:
  - If e forward edge:
    $f'(e) = f'(e) + b$
  - If e backward edge:
    $f'(v, u) = f'(v, u) - b$ (decrease reversed edge)
- **return** f'

Improving Flow

Lemma (Flow Improvement)

Let f be a flow in G with $f_{\text{in}}(s) = 0$ and P an augmenting path with respect to f. If f' is the output from augment(f, P), then f' is a flow with

$$\text{value}(f') = \text{value}(f) + \mathit{bottleneck}(P, f)$$

and $f'_{\text{in}}(s) = 0$.

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- To check that f' is a flow, need to check capacity constraint and flow conservation constraint.

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- Let $b := \text{bottleneck}(P, f)$.
- **Capacity constraint:** given $e \in E(G_f)$, we have
  - e forward edge in G_f, then
  $$f'(e) = f(e) + b \leq f(e) + (c(e) - f(e)) = c(e)$$
  - $e := (u, v)$ backward edge in G_f, then
  $$f'(v, u) = f(v, u) - b \leq f(v, u) \leq c(v, u)$$
  and
  $$f'(v, u) = f(v, u) - b \geq f(v, u) - f(v, u) \geq 0$$

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- Let $b := \text{bottleneck}(P, f)$.
- **Flow Conservation:** let $u \in V$ be a vertex.
  - if $u \notin P$ then flow in and out of u doesn't change.

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- Let $b := \text{bottleneck}(P, f)$.
- **Flow Conservation:** let $u \in V$ be a vertex.
  - if $u \in P$, have 4 cases to analyze. Let $e_1 := (w, u)$ and $e_2 := (u, z)$ be the edges in P passing through u in G_f.
    1. e_1, e_2 forward edges: *both* incoming and outgoing flow *increase* by b
    2. e_1, e_2 backward edges: *both* incoming and outgoing flow *decrease* by b
    3. e_1 forward, e_2 backward: *both* incoming and outgoing flow *unchanged*
    4. e_1 backward, e_2 forward: *both* incoming and outgoing flow *unchanged*

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- Let $b := \text{bottleneck}(P, f)$.
- Value of flow f' and $f'_{\mathrm{in}}(s)$:
  - $f_{\mathrm{in}}(s) = 0 \Rightarrow$ no backward edges incident to s in G_f

$$f'_{\mathrm{in}}(s) = f_{\mathrm{in}}(s) + 0 = f_{\mathrm{in}}(s) = 0$$

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- Let $b := \text{bottleneck}(P, f)$.
- Value of flow f' and $f'_{\mathrm{in}}(s)$:
    - Value of f': by previous bullet, only forward edges out of s, thus:

$$\mathrm{value}(f') = f'_{\mathrm{out}}(s) = f_{\mathrm{out}}(s) + b = \mathrm{value}(f) + b$$

Ford-Fulkerson Algorithm

Now that we know that augmenting paths can only improve our flow, we can describe Ford-Fulkerson, which simply applies the augmenting operation until we can no longer do it.

- Ford-Fulkerson(G):
  1. Initialize $f(e) = 0$ for all $e \in E$, and initialize G_f accordingly
  2. While there is $s \to t$ path $P \in G_f$:
     - $f \leftarrow \text{augment}(f, P)$
     - update G_f
  3. **return** f

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Next lecture: runtime analysis and proof of correctness.

Acknowledgement

Based on

- Prof. Lau's Lecture 15

  https://cs.uwaterloo.ca/~lapchi/cs341/notes/L15.pdf

- Jeff Erickson's book, Chpater 10

  https://jeffe.cs.illinois.edu/teaching/algorithms/book/10-maxflow.pdf

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