Lecture 18: Max-Flow & Min-Cut Applications

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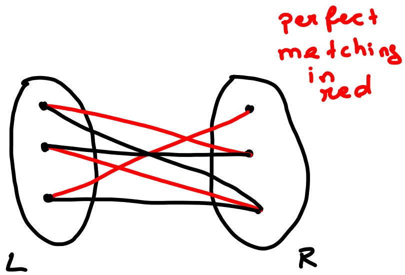
November 14, 2023

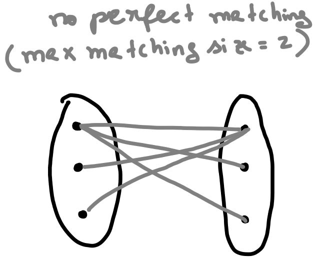
Overview

- Applications of Max-Flow & Min-Cut
  - Maximum Bipartite Matching
  - Minimum Vertex Cover
  - Edge-disjoint Paths
  - Vertex-disjoint Paths
- Further Remarks
- Acknowledgements

Matchings

- Given an undirected graph $G(V, E)$ a *matching* M is a subset of E such that all edges in M are pairwise vertex disjoint (i.e., no two edges share a common vertex)
- A matching $M \subset E$ is called a *perfect matching* if every vertex in the graph is matched.





Maximum Bipartite Matching

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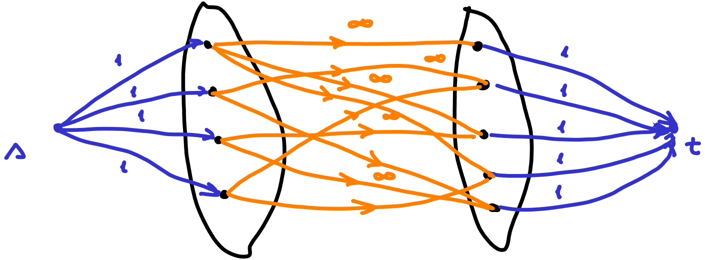
$$\begin{cases} \{u,v\} \in E,\ \ u \in L, v \in R \Leftrightarrow (u,v) \in F,\ \ c(u,v) = \infty \\ (s,u) \in F,\ \ c(s,u) = 1\ \forall\ u \in L \\ (v,t) \in F,\ \ c(v,t) = 1\ \forall\ v \in R \end{cases}$$

in picture:



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Maximum Bipartite Matching

- **Claim:** there is matching of size k in $G \Leftrightarrow$ there is an $s \rightarrow t$ flow of value k in H
  - ($\Rightarrow$) from matching $M = \{\{u_i, v_i\}\}_{i=1}^{k}$ we get flow $f(s, u_i) = f(u_i, v_i) = f(v_i, t) = 1$ of value k

Maximum Bipartite Matching

- **Claim:** there is matching of size k in $G \Leftrightarrow$ there is an $s \to t$ flow of value k in H

  - ($\Leftarrow$) from (integral) flow of value k (exists by Ford-Fulkerson), use flow decomposition lemma (note that H is a DAG) to get k $s \to t$ paths $P_1, \ldots, P_k$, where
$$P_i = (s, u_i, v_i, t)$$
    Path decomposition lemma says that (s, u_i)'s and (v_i, t)'s must be distinct, since
$$0 < f(s, u_i) \leq c(s, u_i) = 1 \Rightarrow f(s, u_i) = 1$$
    (same for (v_i, t)).
    Moreover, $\{u_i, v_i\} \in E$ for $i \in [k]$, by construction of H.
    Thus, $M = \{\{u_i, v_i\}\}_{i=1}^k$ must be a matching in G.

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    Thus, $M = \{\{u_i, v_i\}\}_{i=1}^{k}$ must be a matching in G.
- Ford-Fulkerson gives algorithm with running time $O(|V| \cdot |E|)$ for maximum bipartite matching.

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  - Maximum Bipartite Matching
  - Minimum Vertex Cover
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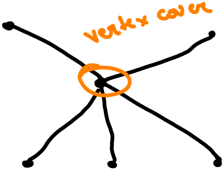
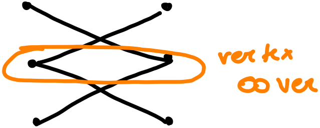
Minimum Vertex Cover

- **Definition:** given graph $G(V, E)$, a subset $S \subseteq V$ is a *vertex cover* if for every edge $\{u, v\} \in E$, we have $\{u, v\} \cap S \neq \emptyset$





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In a bipartite graph, the maximum size of a matching **equals** *the minimum size of a vertex cover.*

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In a bipartite graph, the maximum size of a matching **equals** *the minimum size of a vertex cover.*

- Ford-Fulkerson finds a min-cut in the modified graph H from the previous slides, and from it we will obtain a vertex cover. (we'll see this in the next side)

Proof of König's theorem

- Let $G(L \sqcup R, E)$ be our bipartite graph and k be the maximum size of a matching in it.
- Let $H(\{s, t\} \sqcup L \sqcup R, F)$ be constructed as before. By our previous result, the max-flow in H has value k.

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- By the max-flow min-cut theorem, let S be an $s - t$ cut in H with $s \in S$ & $C_{\mathrm{out}}(S) = k$. (Ford-Fulkerson finds us such cut)

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- **Claim 2:** $(L \setminus S) \cup (S \cap R)$ is a vertex cover of G
  - Note that $\delta_{out}(S)$ cannot contain edge from L to R (as these have ∞ capacity).
  - Thus, every edge in G must be from $L \setminus S$ **or** to $S \cap R \Rightarrow$ vertex cover

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Theorem (Hall's Theorem)

A bipartite graph $G(L \sqcup R, E)$ *with* $|L| = |R| = n$ *has a perfect matching* $\Leftrightarrow$ *for every subset* $S \subset L$*, it holds that* $|N(S)| \geq |S|$.

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- Proof of this theorem can be derived from König's theorem.
  - **Hint:** can we have a vertex cover of size $< n$ when the neighborhood constraints hold?

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- **Claim 3:** there are k edge-disjoint $s \to t$ paths iff there is $s \to t$ flow of value k
  - ($\Rightarrow$) given k edge disjoint paths $P_1, \ldots, P_k$, we can simply get a flow of value k by "adding" the paths P_i, that is, set the flow value to be 1 for each edge in one of the paths, and all other edges get 0 capacity
  - ($\Leftarrow$) given flow of value k, by flow decomposition theorem we have k paths $P_1, \ldots, P_k$, and these must be edge disjoint, since for any $e \in E$, we have $0 \leq f(e) \leq c(e) = 1$.

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- By the max-flow min-cut theorem, can prove:

The maximum number of edge-disjoint $s \to t$ paths equals the minimum number of edges whose removal disconnects s and t (i.e., no $s \to t$ paths).

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- For each $u \in V \setminus \{s, t\}$, replace it by two vertices u_1, u_2 and edges

$$\begin{cases} (u_1, u_2) \\ (w, u_1), \ \forall\ w \in N_{in}(u) \\ (u_2, v), \ \forall\ v \in N_{out}(u) \end{cases}$$

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- **Claim 4:** There are k vertex-disjoint $s \to t$ paths in $G \Leftrightarrow$ there are k edge-disjoint $s \to t$ paths in the new graph.
- In this case, Ford-Fulkerson also gives us a $O(|V| \cdot |E|)$ time algorithm.

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Duality Theorems

- It may at first seem a little magic that vertex cover and matching are dual problems.
- In fact several combinatorial optimization problems have very natural dual problems, and the knowledge of such duality is a powerful algorithmic tool!
- Most (efficient) combinatorial optimization problems captured by

  Linear Programming

  one of the most powerful framework for efficient computation.
- Most of the dual statements seen here can be derived from *Linear Program Duality*
- For more on this topic we encourage you all to take some courses in C&O about it.

Acknowledgement

Based on

- Prof. Lau's Lecture 16

  https://cs.uwaterloo.ca/~lapchi/cs341/notes/L16.pdf

- Jeff Erickson's book, Chapter 11

  https://jeffe.cs.illinois.edu/teaching/algorithms/book/11-maxflowapps.pdf

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