Lecture 20: Reductions II

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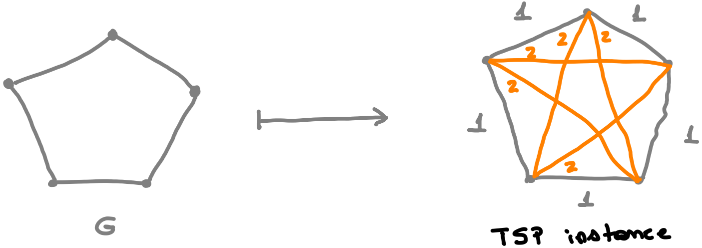
Overview

- More Reductions
  - Hamiltonian Cycle and Traveling Salesman Problem (TSP)
  - SAT, 3SAT & Independent Set
  - Graph Coloring & 3SAT
  - Subset Sum & Vertex Cover
- Web of Reductions
- Acknowledgements

Hamiltonian Cycle & Traveling Salesman Problem (TSP)

- **Claim 1:** hamiltonian cycle $\leq_m$ TSP
- **Reduction:** different edge weights (for edges in graph vs edges not in graph)



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SAT

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- a *literal* is a variable, or its negation (i.e., $x_i, \overline{x_i}$)

SAT

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- **CNF:** a boolean formula is in conjunctive normal form (CNF) if:
  - it is the (conjunction) AND of a number of *clauses*,
  - each clause being an (disjunction) OR of some *literals*

  Example:

$$(x_1 \vee x_3 \vee \overline{x_4}) \ \wedge \ (\overline{x_1} \vee x_2 \vee \overline{x_5} \vee x_6) \ \wedge \ (\overline{x_2} \vee x_4)$$

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- **SAT** problem
  - **Input:** a CNF formula
  - **Output:** YES, if it has a satisfying assignment; NO otherwise

3SAT

- a *3CNF* formula is a CNF formula with *exactly 3 literals* per clause
  Example:

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- **3SAT** problem
  - **Input:** a 3CNF formula
  - **Output:** YES, if it has a satisfying assignment; NO otherwise

- Why are we talking about this problem?

Exercise: prove that SAT $\leq_m$ 3SAT.

3SAT & Independent Set

- **Claim 2:** 3SAT $\leq_m$ IS

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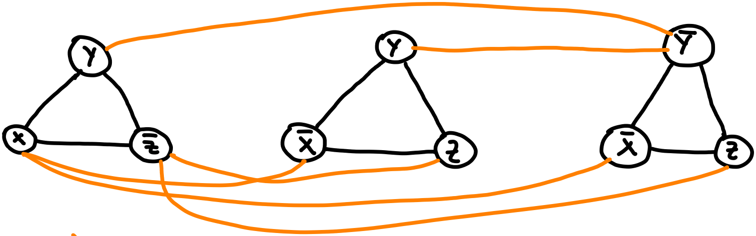
- **Proof:** construct "conflict graph."

Example : $(x \vee y \vee \bar{z}) \wedge (\bar{x} \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee z)$



conflict graph

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Graph Coloring

- Given graph $G(V, E)$ and $k \in \mathbb{N}$, a *proper k-coloring* of G is a function $C : V \to \{1, 2, \ldots, k\}$ such that

$$\text{For all } \{u, v\} \in E \text{ we have } C(u) \neq C(v).$$

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- **3 Coloring** (3COLOR) problem
  - **Input:** graph $G(V, E)$
  - **Output:** does G admit a proper 3-coloring?

3SAT & 3COLOR

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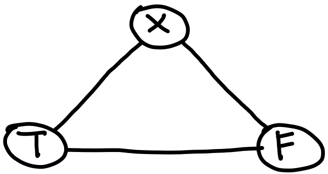
- we use gadgets - subgraphs enforcing semantics of input formula φ
  - *Truth gadget*: triangle with 3 vertices T, F, X (standing for True, False, Other)

    Enforces that T will be assigned color True, F will be assigned color False and X will be assigned the third color



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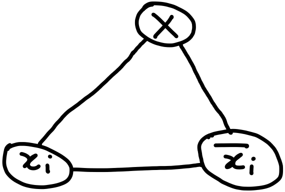
  - *Literal gadget*: for each $x_i, \overline{x_i}$, we have a triangle with vertices $X, x_i, \overline{x_i}$

    Enforces x_i and $\overline{x_i}$ get a proper assignment.



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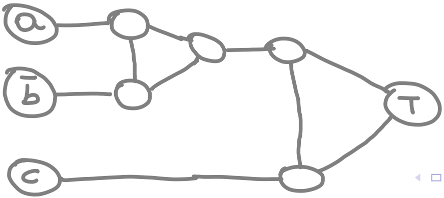
    Enforces x_i and $\overline{x_i}$ get a proper assignment.
  - *Clause gadget*: enforces each clause that becomes true under assignment will have a 3 coloring (iff)

Clause: $a \vee \bar{b} \vee c$



3SAT & 3COLOR - correctness

Need to prove following claims:

- Literal gadget enforces every variable is properly assigned in a coloring
- Clause gadget enforces that every valid 3-coloring of the graph corresponds to a variable assignment which makes corresponding clause true

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- **Claim 4:** Vertex Cover $\leq_m$ Subset Sum

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- **Proof:** given $G(V,E)$ and k, need to construct (in poly-time) a (multi)set X of integers and T such that:

  X has a subset that sums to $T \Leftrightarrow G$ has a vertex cover of size k.

Subset Sum & Vertex Cover

- **Claim 4:** Vertex Cover $\leq_m$ Subset Sum
- **Proof:** given $G(V, E)$ and k, need to construct (in poly-time) a (multi)set X of integers and T such that:

  X has a subset that sums to $T \Leftrightarrow G$ has a vertex cover of size k.
- Reduction:
  - Number edges (arbitrarily) from 0 to $m-1$. Edge i will correspond to integer $b_i := 4^i$
  - For each vertex $u \in V$ assign number

    $$a_u := 4^m + \sum_{i \in \delta(u)} 4^i$$

    where $\delta(u)$ is the set of edges with u as one endpoint.
  - Let

    $$T := k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i$$

  - Let $X = \{a_u, b_i\}_{u \in V, 0 \leq i < m}$

Subset Sum & Vertex Cover - proof of reduction

- Need to prove that $\langle G, k \rangle$ has a vertex cover of size k iff X has a subset of elements with sum T

Subset Sum & Vertex Cover - proof of reduction

- Need to prove that $\langle G, k \rangle$ has a vertex cover of size k iff X has a subset of elements with sum T
- ($\Rightarrow$) Let $C \subset V$ be a vertex cover of size k. Consider the subset

$$Y := \{a_u \mid u \in C\} \cup \{b_i \mid \text{edge } i \text{ has exactly one endpoint in } C\}$$

  easy to check it has sum T

Subset Sum & Vertex Cover - proof of reduction

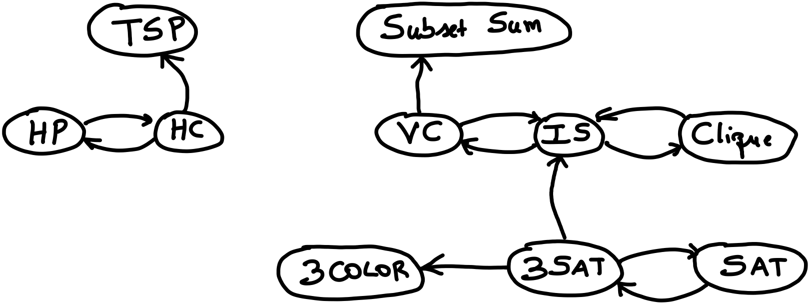
- Need to prove that $\langle G, k \rangle$ has a vertex cover of size k iff X has a subset of elements with sum T

- ($\Leftarrow$) Let $\{a_u\}_{u \in C} \cup \{b_i\}_{i \in F} =: Y \subset X$ be a subset with sum T. Must have:

$$\sum_{u \in C} a_u + \sum_{i \in F} b_i = T = k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i$$

  since there are no carries from lower order base-4 digits (i.e., the b_i's), it must be the case that $|C| = k$. moreover, to each 4^i, there is at most one b_i on LHS that contributes with 4^i, so C must be a vertex cover.

Current algorithmic world view



- Wait, why haven't we proved the missing arrows? Do they even hold?

Foundational Question

- Is there a way to organize our world view?
  - Is there some property that unifies the problems we have seen so far?
  - Why would any of these be considered "hard"?
  - Can we "classify" problems according to their "difficulty"? How can we measure this?

Acknowledgement

Based on

- [Kleinberg Tardos 2006, Chapter 8]
- [Erickson 2019, Chapter 12]

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