Lecture 21: Intractability - NP and coNP

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

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Overview

- Complexity Classes & Complete Problems
  - NP
  - coNP
  - Completeness for NP
- Completing Karp Reductions/Polynomial Transformations
  - NP-completeness of 3SAT
  - Current Worldview
- Acknowledgements

NP

- Let Π be a decision problem and let L_Π be the set of all YES instances of Π. Then $L_\Pi \subseteq \{0,1\}^*$

  decision problems $\leftrightarrow$ subsets of all boolean strings

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- NP := class of decision problems Π with following property:
  - There is a poly-time algorithm V_Π and a constant $c > 0$ such that
    - For any $x \in L_\Pi$ (i.e., YES instance) of size n, there is a *proof/witness* y of size n^c such that $V_\Pi(x, y) = 1$
    - For any $x' \notin L_\Pi$ (i.e., NO instance) there is *no such proof* z of size n^c such that $V_\Pi(x', z) = 1$.

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- In other words, NP is the class of decision problems where the YES instances have a *small proof* that can be *verified* in poly-time

Problems in NP

- Clique
- Independent Set
- SAT (and 3SAT)
- TSP
- Hamilton cycle (and Hamilton path)
- Subset Sum
- Vertex Cover
- 3COLOR (and the graph coloring problem)
- *every* problem in P

coNP

- The class coNP is essentially the opposite of NP.
- for a decision problem Π, let $\overline{\Pi}$ be the *opposite* problem to Π, that is,

$$x \in L_{\Pi} \Leftrightarrow x \notin L_{\overline{\Pi}}$$

  equivalently, $L_{\overline{\Pi}} = \overline{L_{\Pi}}$.
- In simpler terms, every YES instance of Π is a NO instance of $\overline{\Pi}$ (and vice-versa)

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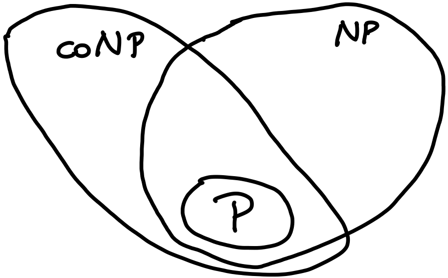
- In simpler terms, every YES instance of Π is a NO instance of $\overline{\Pi}$ (and vice-versa)
- coNP := class of decision problems Π such that $\overline{\Pi} \in$ NP.

Relation between P, NP and coNP



Unknown:

1) is $P = NP \cap coNP$?
2) is $NP = coNP$?
3) is $P = NP$?

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A remark about reductions

- Given a particular reduction $\leq$ (Turing, Karp), we can define a *complete* problem for a complexity class $\mathcal{C}$ as follows:
  - **Hardness:** Π is $\mathcal{C}$-hard if for *every* problem $\Gamma \in \mathcal{C}$, we have

$$\Gamma \leq \Pi$$

  - **Membership in $\mathcal{C}$:** $\Pi \in \mathcal{C}$

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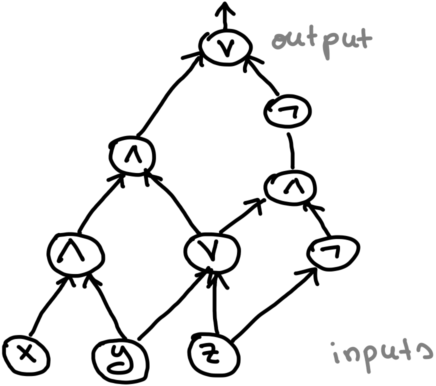
  - **Membership in $\mathcal{C}$:** $\Pi \in \mathcal{C}$
- Complexity theorists prefer to define NP-completeness under *Karp reductions* (or polynomial transformations) because, as we will see, NP is closed under such reductions
  - Note that we *do not know* whether NP is closed under Turing reductions
  - The above would imply NP $=$ coNP, which is considered unlikely

$$\text{Under Turing reductions, UNSAT} \equiv \text{SAT}$$

CIRCUIT-SAT

- A *boolean circuit* is a DAG with:
  - input gates
  - AND/OR/NOT gates,
  - and a special gate (the output gate)



CIRCUIT-SAT

- A *boolean circuit* is a DAG with:
  - input gates
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  - and a special gate (the output gate)
- **CIRCUIT-SAT** problem:
  - **Input:** a boolean circuit Φ
  - **Output:** YES, if there is a truth assignment α such that $\Phi(\alpha) = 1$, NO otherwise.

Cook-Levin Theorem: CIRCUIT-SAT is NP-complete

Theorem (Cook-Levin)

CIRCUIT-SAT is NP-complete under polynomial transformations.

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- Want to prove that for any $\Pi \in \mathrm{NP}$, we have $\Pi \leq_m$ CIRCUIT-SAT
- **Proof sketch:** *computation is local*
  - $\Pi \in \mathrm{NP} \Rightarrow \exists$ poly-time verification algorithm V_Π and $c > 0$ such that for any instance $x \in \{0,1\}^n$,

$$x \in L_\Pi \Leftrightarrow \exists y \in \{0,1\}^{n^c} \text{ s.t. } V_\Pi(x,y) = 1$$

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  - If $V_\Pi(x,y)$ runs in time $O(n^\gamma)$ (since it is polynomial in terms of the input size), there is circuit of size $O(n^\gamma)$ simulating computation of V_Π

    Can construct this circuit (from description of V_Π) in $\mathrm{poly}(n)$-time!

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  - So, we get a $\mathrm{poly}(n)$-sized circuit $\Phi_x(y)$ which is satisfiable iff $x \in L_\Pi$!

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  - So, we get a $\mathrm{poly}(n)$-sized circuit $\Phi_x(y)$ which is satisfiable iff $x \in L_\Pi$!
  - Thus, we have a transformation

$$x \mapsto \Phi_x$$

  such that $x \in L_\Pi \Leftrightarrow \Phi_x \in$ CIRCUIT-SAT.

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- To prove this, by Cook-Levin theorem, need to show that

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- Let $\Phi \in$ CIRCUIT-SAT of size n (i.e., n gates and wires). We will construct CNF Ψ with $O(n)$ clauses such that

$$\Phi \text{ is satisfiable} \Leftrightarrow \Psi \text{ is satisfiable.}$$

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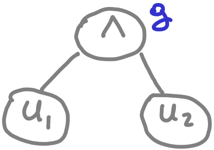
$$\Phi \text{ is satisfiable} \Leftrightarrow \Psi \text{ is satisfiable.}$$
- Can do the above simulating gate-by-gate (wire-by-wire):
  - each gate has a new variable, which will tell us the value of the gate
  - Simulate each gate operation (AND/OR/NOT) as a CNF
  - ensure that output gate variable should be true

Gate Simulations

- **AND:** CNF

$$(\overline{g} \vee u_1) \wedge (\overline{g} \vee u_2) \wedge (g \vee \overline{u_1} \vee \overline{u_2})$$



$\longmapsto$

$(\bar{g} \vee u_1) \wedge (\bar{g} \vee u_2)$

if u_1 or u_2 $= 0$ then must have $g = 0$ to satisfy above

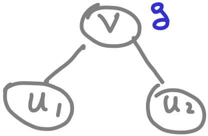
$(g \vee \bar{u}_1 \vee \bar{u}_2)$

if both $u_1 = u_2 = 1$ then $g = 1$

Gate Simulations

- **OR:** CNF

$$(g \vee \overline{u_1}) \wedge (g \vee \overline{u_2}) \wedge (\overline{g} \vee u_1 \vee u_2)$$



$(g \vee \bar{u}_1) \wedge (g \vee \bar{u}_2)$

if u_1 or $u_2 = 1$ then g must be 1 to satisfy above

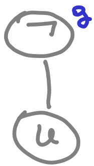
$(\bar{g} \vee u_1 \vee u_2)$

if $u_1 = u_2 = 0$ then g must be 0 to satisfy above clause.

Gate Simulations

- **NOT**: CNF

$$(\overline{g} \vee \overline{u}) \wedge (g \vee u)$$



$\bar{g} \vee \bar{u}$

if $u = 1$ then $g = 0$ to satisfy above clause

$g \vee u$

if $u = 0$ then $g = 1$ to satisfy above clause

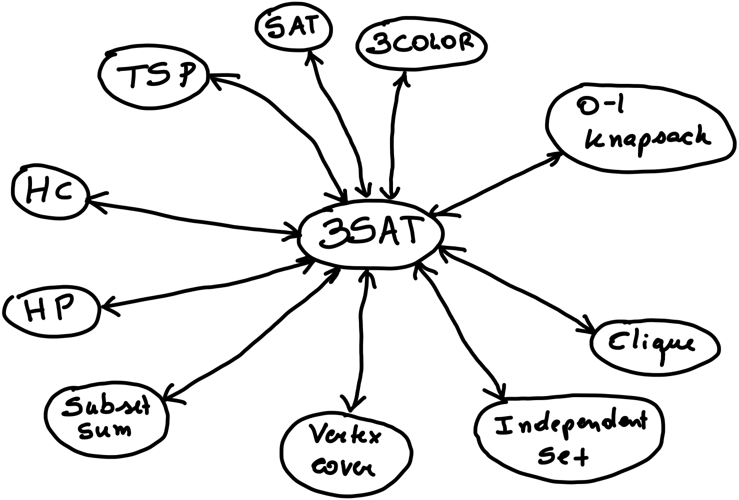
Gate Simulations

- **1**: CNF is simply literal g

Gate Simulations

- $\mathbf{0}$: CNF is simply literal $\overline{g}$

Updated Worldview



All NP-complete problems!

Where do we go next ?

- CS 360/365

Formalization of Algorithms , full proof of Cook-Levin & much more!

(Prof Blais teaching it nex term)

- Are there harder problems ?

For sure! See CS 360/365 or more advanced courses

Acknowledgement

Based on

- [Erickson 2019, Chapter 12]
- Prof. Lau's Lecture 18 notes

  https://cs.uwaterloo.ca/~lapchi/cs341/notes/L18.pdf

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