CS 341: Algorithms
Module 3: Reductions, Recurrences

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Based on lecture notes by many previous CS 341 instructors

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Spring 2019
The 2SUM problem

Problem:
Given an array $A[1..n]$ of integers and integer $m$, find if there are indices $i$ and $j$ (not necessarily distinct!) such that $A[i]+A[j] = m$.

First solution
Algorithm 1: $\text{SIMPLE2SUM}(A[1..n], m)$

1. for $i = 1$ to $n$ do
2.     for $j = i$ to $n$ do
3.         if $A[i]+A[j] = m$ then
4.             return true
5. return false

Analyze this ... $\Theta(n^2)$
Second solution

Algorithm 2: `Faster2SUM(A[1..n], m)`

1. Sort(A)
2. for $i = 1$ to $n$ do
3.  $j = \text{BinarySearch}(m-A[i], A)$
4.  if $j > 0$ then
5.    return true
6. return false

Analyze this ... $\Theta(n \log n)$
Third solution (Assuming array is already sorted by Sort(A));

Algorithm 3: \textsc{Sorted2Sum}(A[1..n], m)

\begin{algorithm}
\begin{algorithmic}[1]
\STATE $i = 1; j = n$;
\WHILE{$i \leq j$}
\IF{$A[i] + A[j] = m$}
\STATE return true
\ELSE\IF{$A[i] + A[j] < m$}
\STATE $i = i+1$
\ELSE
\STATE $j = j-1$
\ENDIF
\ENDIF
\ENDWHILE
\STATE return false
\end{algorithmic}
\end{algorithm}

Analyze this ... $\Theta(n \log n)$, but the second stage (\textsc{Sorted2Sum}) is $\Theta(n)$. 
Sketch of correctness proof:

If there is no such pair of indices, our algorithm will not find it (as we will never get true at line 3).
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Suppose there is such pair $i' \leq j'$ such that $A[i'] + A[j'] = m$, we only need to prove that our algorithm will not miss this pair. Without loss of generality we suppose $i$ becomes $i'$ when $j > j'$. The other direction can be proved by symmetry.
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Since array is sorted, $A[j] \geq A[j']$. So, the if-else branch will not increase $i$ anymore. It will keep reducing $j$ until finds $j'$ pair.
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For each value of $k$, use `FASTER2SUM` (second solution) to find $i, j : A[i] + A[j] = -A[k]$ ($\Theta(n^2 \log n)$).
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This is the reduction technique in algorithm design.
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Solution 3:
Pre-sort A to avoid sorting it for each $k$, still use Faster2SUM idea (binary search).
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Solution 4: Use a better Sorted2SUM solution (on array sorted once)!

Algorithm 5: $\text{Fast3SUM}(A[1..n], m)$

1. Sort(A);
2. for $k = 1$ to $n$ do
3. $x = \text{Sorted2SUM}(A,-A[k])$
4. if $x$ then
5. return $true$
6. return $false$

Analyze this ... $\Theta(n^2)$
The mergesort recurrence is

\[
T(n) = \begin{cases} 
T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + \Theta(n) & \text{if } n > 1 \\
\Theta(1) & \text{if } n = 1.
\end{cases}
\]
Recurrence Relations

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It is simpler to consider the following exact recurrence, with constant factors \( c \) and \( d \) replacing \( \Theta \)'s:

\[ T(n) = \begin{cases} 
T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + cn & \text{if } n > 1 \\
d & \text{if } n = 1.
\end{cases} \]
The following is the corresponding *sloppy* recurrence (it has floors and ceilings removed):

\[ T(n) = \begin{cases} 
2 \ T \left( \frac{n}{2} \right) + cn & \text{if } n > 1 \\
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The exact and sloppy recurrences are identical when \( n \) is a power of 2.

We will begin by solving the sloppy recurrence when \( n = 2^j \) using the recursion tree method.
Recursion Tree Method

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3. Repeat this process recursively, terminating when a node receives the value \( T(1) = d \).
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3. Repeat this process recursively, terminating when a node receives the value $T(1) = d$.

4. Sum the values on each level of the tree, and then compute the sum of all these sums; the result is $T(n)$. 
Solving the Exact Recurrence

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If this solution is expressed as a function of \( n \) using \( \Theta \)-notation, then we obtain the complexity of the solution of the exact recurrence for all \( n \).

This is not a proof, however. If a real mathematical proof is required, then it is necessary to use induction.
The Master Method

The “Master Theorem” provides a formula for the solution of many recurrence relations typically encountered in the analysis of divide-and-conquer algorithms.
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The following is a simplified version (a more general version can be found in the textbook):

Theorem (Master theorem)

Suppose that $a \geq 1$ and $b > 1$. Consider the recurrence

$$T(n) = a \cdot T \left( \frac{n}{b} \right) + \Theta(n^y)$$

in sloppy or exact form. Denote $x = \log_b a$. Then

$$T(n) \in \begin{cases} 
\Theta(n^x) & \text{if } y < x \\
\Theta(n^x \log n) & \text{if } y = x \\
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\end{cases}$$
The Master Method

Suppose that $a \geq 1$ and $b \geq 2$ are integers and

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^y, \quad T(1) = d.$$ 

Let $n = b^j$. 

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<thead>
<tr>
<th>size of subproblem</th>
<th># nodes</th>
<th>cost/node</th>
<th>total cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = b^j$</td>
<td>1</td>
<td>$c \cdot n^y$</td>
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<tr>
<td>$n/b = b^{j-1}$</td>
<td>$a$</td>
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Computing $T(n)$

Summing the costs of all levels of the recursion tree, we have that

$$T(n) = d \ a^j + c \ n^y \sum_{i=0}^{j-1} \left( \frac{a}{b^y} \right)^i.$$
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The formula for $T(n)$ is a geometric sequence with ratio $r = \frac{a}{b^y} = b^{x-y}$:

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$$T(n) = d \cdot n^x + c \cdot n^y \sum_{i=0}^{j-1} r^i.$$

There are three cases, depending on whether $r > 1$, $r = 1$ or $r < 1$. 
## Complexity of $T(n)$

<table>
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<tr>
<th>Case</th>
<th>$r$</th>
<th>$y, x$</th>
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"Heavy leaves" means that cost of the recursion tree is dominated by the cost of the leaf nodes.

"Balanced" means that costs of the levels of the recursion tree stay "almost the same" (except for the last level).

"Heavy top" means that cost of the recursion tree is dominated by the cost of the root node.
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**Theorem**

*Suppose that \( a \geq 1 \) and \( b > 1 \). Consider the recurrence*

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\]

*Denote \( x = \log_b a \). Then*

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T(n) \in \begin{cases} 
\Theta(n^x) & \text{if } y < x \\
\Theta(n^y \log^{m+1} n) & \text{if } y = x \\
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