CS 341: Algorithms
Module 7: Graph Algorithms

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Graphs and their representation:

- A graph $G = (V, E)$ consists of a set of vertices $V = \{v_i \mid i = 1, 2, \cdots, n\}$ and a set of edges $E = \{e_j \mid j = 1, 2, \cdots, m\}$ that connect the vertices.
  - An edge $e$ may be represented by the pair $(u, v)$ where $u$ and $v$ are the vertices being connected by $e$.
    - Depending on the problem, the pair may be ordered or unordered.

- Many problems can be represented as graphs:
  - Traveling Salesman Problems
  - Airline flights
  - Friends who know each other
  - Moves in a game (each node = the state of the board or game).
Directed and Undirected Graphs

- An undirected graph, $G = (V, E)$ is a pair:
  - $V =$ set of distinct vertices.
  - $E =$ set of edges; each member is a set of 2 vertices
    - For example:
      
      $$V = \{t, u, v, w, x, y, z\}$$
      $$E = \{\{u, v\}, \{u, w\}, \{v, w\}, \{v, y\}, \{x, z\}\}$$

- A directed graph, $G = (V, E)$:
  - Is the same as an undirected graph except $E$ is a set of ordered pair:
    - For example:
      
      $$E = \{(a, b), (a, c), (c, b), (b, e), (e, b)\} \quad (1)$$
More Definitions

- **Weighted graphs:**
  - A weighted graph has value assigned to each of its edges.
  - More formally, there is a weight function $w : E \to R$.
    - $R = \text{real numbers}$.
    - Depending on the syntax being used, we may see this as $w(e)$ with $e \in E$ or $w(u, v)$ with $u$ and $v$ representing the vertices for the edge.

- **Degree:**
  - The degree of vertex $v$, denoted by $\text{deg}(v)$, is the number of edges that meet at $v$.
  - Let $v$ be a vertex in a directed graph $G$, The number of vertices of $G$ adjacent **from** $v$ is called the outdegree of $v$. the number of vertices of $G$ adjacent **to** $v$ is called the indegree of $v$. 
**Representations of Graphs**

- **Adjacency matrix:**
  - $M[i, j] = 1$ if $i$ and $j$ are neighbours, 0 otherwise.
  - Assign each vertex an integer index (in this example, $t = 7$, $u = 2$, etc.)
  - Assumes that any other info for a vertex and/or edge is in another data structure.
  - The matrix is symmetric for an undirected graph.
  - For a directed graph we will likely have an asymmetric matrix.

```
   u  v  w  x  y  z  t
  ____  ____  ____  ____  ____  ____  ____
   u  1  1  ____  ____  ____  ____  ____
   v  1  ____  1  1  ____  ____  ____
   w  1  1  ____  ____  ____  ____  ____
   x  ____  ____  ____  ____  1  ____  ____
   y  ____  ____  ____  ____  ____  ____  1
   z  ____  ____  ____  ____  ____  ____  ____
   t  ____  ____  ____  ____  ____  ____  ____
```
Representations of Graphs

- **Adjacency matrix notes:**
  - Blank row: no neighbours i.e. isolated vertex.
  - \( M[i, i] = \) self-loop.
  - Undirected graphs are symmetrical

- **Space**
  - \(|V|^2\) bits
  - \((|V|^2 + |V|)/2\) (if undirected, but harder to actually implement).
  - Additional information, such as cost of an edge, could be stored in the matrix. Another option is to store a pointer to this information.

- **Cost of operations**
  - Are vertices \( i \) and \( j \) adjacent? \( O(1) \)
  - Add or delete edge: \( O(1) \)
  - Add vertex: increases size of matrix.
  - Find neighbours of \( v \): \( O(|V|) \).
Representations of Graphs

- Adjacency list notes:
  - Space:
    - $O(|V| + |E|)$
    - Usually much smaller than $O(|V|^2)$ for a sparse graph.
- Cost of operations:
  - Add an edge $O(1)$
  - Delete an edge: Search lists for each endpoint $O(|V|)$.
  - Add vertex: Depends on the implementation.
  - Find neighbours $O$(number of neighbours) (better than Adj. Matrix)
  - Are $i,j$ adjacent? Search the list (worse than Adj. Matrix)
More Notation

- **Path:**
  - Given the graph $G(V, E)$, a path from vertex $u$ to vertex $v$ is a sequence of vertices $(v_0, v_1, \cdots, v_k)$ such that $v = v_0$, $u = v_k$, and $(v_{i-1}, v_i)$ is in $E$ for all $i = 1, 2, \cdots, k$.

- **Simple path:**
  - A path is simple if all the vertices in the path are distinct.

- **Cycle:**
  - A path $(v_0, v_1, \cdots, v_k)$ forms a cycle if $v_0 = v_k$, and the path contains at least one edge.
  - A graph with no cycles is acyclic.

- **Connected graphs:**
  - An undirected graph is connected if every pair of vertices is connected by a path.
BFS and DFS

- Breadth-first, depth-first search
- Each starts at an arbitrary node, explores the whole connected component
- Assume whole graph is connected
- General view: vertices start out coloured white (not visited)
- A visited vertex is coloured gray (visited, but may have white neighbours)
- When all neighbours of a vertex are visited, it is coloured black.
General view of searching

- Gray nodes form a “frontier”
- Can choose any neighbour of a gray node to be next visited
- In general, want to perform computation
  - Preprocess when colouring gray
  - Postprocess when colouring black
  - Analogous to tree traversal uses
Breadth-first search

- Use queue (first-in, first-out) to store gray nodes
- Start by taking any vertex, colouring it gray, add it to queue
- Repeat: find white node adjacent to head of queue, colour it gray and add it to queue
- When you can’t find any such node, remove the head of the queue and colour it black
Pseudocode for BFS

colour_all_vertices_white()
while there is a white vertex s do
    BFS_tree(s)

BFS_tree(s: vertex)
colour s gray (visited)
enqueue(Q,s)
while Q not empty do
    u <- dequeue(Q)
    for each v adjacent to u
        if v white then
            colour v gray
            enqueue(Q,v)
        \( (u,v) \) is a tree edge
        else
            \( (u,v) \) is non-tree edge
    colour u black
Analysis of BFS

- Assume adjacency list representation
- Each vertex enqueued once (colour changes from white to gray) and dequeued once (colour changes from gray to black)
- Body of inner while loop takes $\Theta(1)$ time
- Inner while loop implemented by scanning adjacency list of head of queue
- Therefore each edge looked at exactly twice
- Running time is $\Theta(|V| + |E|)$ or $\Theta(n + m)$
BFS trees

- BFS finds a spanning tree of the graph (edges representing first visits)
- Nontree edges are called cross edges
- A cross edge cannot connect a node to its ancestor in the tree
- A cross edge cannot connect a node to its descendant in the tree
- These are useful in proving properties of BFS searches
Single-source shortest path

- Can use BFS to compute distances $\delta(s, v)$ from source $s$ to all other vertices $v$
- Compute quantity $d[v]$ in following way
  - $d[s] \leftarrow 0$
  - When adding $v$ to queue because $u$ is at head and there is an edge $(u, v)$, set $d[v] \leftarrow d[u] + 1$
Pseudocode for shortest path

colour_all_vertices_white()
while there is a white vertex s do
    BFS_tree(s)

BFS_tree(s: vertex)
colour s gray (visited)
enqueue(Q,s); d[s] <-- 0
while Q not empty do
    u <-- dequeue(Q)
    for each v adjacent to u
        if v white then
            colour v gray
            enqueue(Q,v); d[v] <-- d[u]+1
            \( (u,v) \) is a tree edge
        else
            \( (u,v) \) is non-tree edge
        colour u black
Why does this work?

**Lemma 1**
For any edge \((u, v)\), we have \(\delta(s, v) \leq \delta(s, u) + 1\).

**Lemma 2**
For all \(v\), \(d[v] \geq \delta(s, v)\).

Proof: By induction on number of enqueues
At beginning, \(d[s] = 0 = \delta(s, s)\) Suppose \(u\) is being visited and adjacent white \(v\) discovered.

\[\delta(s, v) \leq \delta(s, u) + 1 \leq d[u] + 1 = d[v].\]
Why does this work?

Lemma 2 proves $d[v] \geq \delta(s, v)$.

**Lemma 3**
At any point, the $d$-values of vertices in the queue are either $i$ or $i + 1$ for some $i$, and all the $i$ values are in front of all the $i + 1$ values.

```
  i+1  ...  i+1    i    ...    i    i    i
```

front         u
Why does this work?

Proof: By induction on number of queue operations
True at beginning (only one item in queue)
True after \( u \leftarrow \text{dequeue}(Q) \)
True after \( \text{enqueue}(Q, v) \), because \( d[v] = d[u] + 1 \).

Corollary 4

\( d \) values are assigned in increasing order.
Correctness continued

**Theorem 5**

\[ d[v] = \delta(s, v) \]

Let \( v \) be closest vertex to \( s \) with wrong \( d \);
\[ d[v] > \delta(s, v) \]

Let \( u \) be vertex just before \( v \) on shortest \( s - v \) path
\[ \delta(s, v) = \delta(s, u) + 1, \quad d[u] = \delta(s, u), \text{ so } d[v] > d[u] + 1. \]

What colour is \( v \) when \( u \) is dequeued?

- White? Then it should have been visited from \( u \)
- Black? Then \( u \) should have been visited from \( v \)
- Gray? Then it was visited from some \( w \), \( d[v] = d[w] + 1 \), and \( d[w] \leq d[u] \), which implies \( d[v] \leq d[u] + 1 \) contradicting above inequality.

Thus no such \( v \) exists.
BFS application

- Connected components via BFS ...
- Presence of cycles ...
- Bi-partite graphs ...
- ...

...