The "exact" Mergesort recurrence is given on # 81. We will let $d = 0$ (which makes sense if we are counting the number of comparisons) and consider the recurrence

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + T\left(\left\lceil \frac{n}{2} \right\rceil \right) + cn \quad \text{if } n > 1$$

$$T(1) = 0,$$

where $c$ is a positive constant.

We want to prove that $T(n)$ is $O(n \log n)$.

**Giving a Complete Proof from First Principles**

We will find constants $k, n_0 > 0$ such that

$$T(n) \leq kn \log_2 n$$

for all $n \geq n_0$.

First, let’s consider the base case, $n = 1$. We have $T(1) = 0$ and $k \times 1 \times \log_2 1 = 0$, so any $k$ satisfies (1) for the base case.

Now we make an induction assumption that (1) is satisfied for $1 \leq n \leq m - 1$, where $m \geq 2$. We want to prove that (1) is satisfied for $n = m$. Note that $k$ is unspecified so far; we will determine an appropriate value for $k$ as we proceed.

We have

$$T(m) = T\left(\left\lfloor \frac{m}{2} \right\rfloor \right) + T\left(\left\lceil \frac{m}{2} \right\rceil \right) + cm,$$

so it follows that

$$T(m) \leq k \left\lfloor \frac{m}{2} \right\rfloor \log_2 \left\lfloor \frac{m}{2} \right\rfloor + \frac{m}{2} + \frac{m}{2} \log_2 \frac{m}{2} + cm,$$

(2)
by applying the induction hypothesis for \( n = \left\lfloor \frac{m}{2} \right\rfloor \) and \( n = \left\lceil \frac{m}{2} \right\rceil \). Using the facts that \( \left\lfloor \frac{m}{2} \right\rfloor \leq \frac{m}{2} \) and \( \left\lceil \frac{m}{2} \right\rceil \leq m \), we obtain the following from (2):

\[
T(m) \leq k \left\lfloor \frac{m}{2} \right\rfloor \log_2 \frac{m}{2} + k \left\lfloor \frac{m}{2} \right\rceil \log_2 m + cm
\]

\[
= k \left\lfloor \frac{m}{2} \right\rfloor (\log_2 m - 1) + k \left\lceil \frac{m}{2} \right\rceil \log_2 m + cm
\]

\[
= k \log_2 m \left( \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil \right) + cm - k \left\lfloor \frac{m}{2} \right\rceil
\]

\[
= km \log_2 m + cm - k \left\lfloor \frac{m}{2} \right\rceil.
\]

In the last line, we are using the fact that \( \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil = m \). Recall that we are trying to prove that \( T(m) \leq km \log_2 m \). Therefore we will be done provided that

\[
cm - k \left\lfloor \frac{m}{2} \right\rceil \leq 0.
\]

This is equivalent to

\[
k \geq \frac{cm}{\left\lfloor \frac{m}{2} \right\rceil}.
\]

Since \( k \) is required to be a constant, we must find an upper bound for the expression on the right side of this inequality. It is not hard to show that

\[
\frac{m}{\left\lfloor \frac{m}{2} \right\rceil} \leq 3
\]

for all integers \( m \geq 2 \). Therefore we can take \( k = 3c \) and we will have \( T(m) \leq km \log_2 m \), as desired.

By the principle of mathematical induction, we have proven that \( T(n) \leq 3n \log_2 n \) for all \( n \geq 1 \) (so \( n_0 = 1 \)).

**Remark:** Suppose the recurrence instead had the form

\[
T(n) = T \left( \left\lfloor \frac{n}{2} \right\rceil \right) + T \left( \left\lceil \frac{n}{2} \right\rceil \right) + cn \quad \text{if } n > 1
\]

\[
T(1) = d,
\]

where \( c, d > 0 \). In this situation, we could not use \( n = 1 \) as the base case for the induction (why?). It turns out that we would have to take \( n_0 = 2 \) and start with \( n = 2 \) and \( n = 3 \) as base cases for the induction. We leave the details as an exercise.
An Alternate Approach

We describe another approach that actually turns out to be a bit easier. It follows two steps: step 1 is to prove that $T(n)$ is monotone decreasing, which is an “easy” induction. Step 2 is to make use of the exact solution to $T(n)$ when $n$ is a power of two that we computed on slide # 66.

Step 1

Using the exact recurrence, we first prove (by induction) that $T(n) \leq T(n+1)$ for all positive integers $n$.

We first observe that $T(1) = 0$ and $T(2) = 2$, so this inequality holds for $n = 1$.

As an induction assumption, suppose that $T(n) \leq T(n+1)$ holds for $1 \leq n \leq m - 1$, where $m \geq 2$. We want to prove that $T(m) \leq T(m+1)$.

First, suppose that $m$ is even. Then

$$T(m) = 2T\left(\frac{m}{2}\right) + cm$$

and

$$T(m+1) = T\left(\frac{m}{2}\right) + T\left(\frac{m}{2} + 1\right) + c(m+1).$$

Thus

$$T(m+1) - T(m) = T\left(\frac{m}{2} + 1\right) - T\left(\frac{m}{2}\right) + c.$$

We have

$$T\left(\frac{m}{2} + 1\right) - T\left(\frac{m}{2}\right) \geq 0$$

by induction, so it follows that $T(m) \leq T(m+1)$.

The case where $m$ is odd is similar. We have

$$T(m) = T\left(\frac{m-1}{2}\right) + T\left(\frac{m+1}{2}\right) + cm$$

and

$$T(m+1) = 2T\left(\frac{m+1}{2}\right) + c(m+1).$$

Thus

$$T(m+1) - T(m) = T\left(\frac{m+1}{2}\right) - T\left(\frac{m-1}{2}\right) + c.$$

We have

$$T\left(\frac{m+1}{2}\right) - T\left(\frac{m-1}{2}\right) \geq 0$$

by induction, so it follows that $T(m) \leq T(m+1)$.
Step 2

Now, consider an arbitrary positive integer $n$. Then $2^{j-1} < n \leq 2^j$ for some integer $j$. From this and the monotonicity of $T(n)$, we have

$$T(n) \leq T(2^j) = cj2^j < c(2n) \log(2n) = 2cn(\log n + 1).$$

In the above, we are applying the formula proven on slide # 66 after setting $d = 0$, namely, $T(2^j) = cj2^j$. We are also using the fact that $2^j < 2n$. 