1. We will prove this by induction on $k$. First, for our base case, let $k = 0$. Then,

$$u(vu)^0 = u = (uv)^0u.$$ 

Now, let $k \geq 1$ and suppose that $u(vu)^\ell = (uv)^\ell u$ for all $\ell < k$. Then we have

$$u(vu)^k = uvu(vu)^{k-1} = uv(u(vu)^{k-1}) = uv(uv)^{k-1}u = (uv)^k u.$$ 

2. (a) The DFA $A$ is defined $A = (\{q_0, q_1\}, \{0, 1\}, \delta, q_0, \{q_0\})$ where the transition function $\delta$ is specified by

$$\delta(q_0, 0) = q_1 \quad \delta(q_0, 1) = q_0$$
$$\delta(q_1, 0) = q_0 \quad \delta(q_1, 1) = q_1$$

(b) We claim that the DFA $A$ recognizes the language

$$L = \{w \in \{0, 1\}^* \mid |w|_0 = 0 \text{ mod } 2\}.$$ 

That is, $A$ recognizes the language of binary strings with an even number of 0s.

We will prove this by showing that a word $w$ reaches state $q_i$ if and only if $|w|_0 = i \text{ mod } 2$. We will show this by induction on the length of $w$. First, for the base case, let $w = \varepsilon$. Then $\delta(q_0, w) = \delta(q_0, \varepsilon) = 0$ and $|\varepsilon|_0 = 0 = 0 \text{ mod } 2$.

Now let $|w| > 0$ and assume that for every word $w'$ with $|w'| < |w|$, $\delta(q_0, w') = q_i$ if and only if $|w'|_0 = i \text{ mod } 2$. Let $w = w'a$ for a word $w' \in \{0, 1\}^*$ and $a \in \{0, 1\}$. Since $|w'| = k - 1$, by our inductive hypothesis, we have $\delta(q_0, w') = q_i$ if and only if $|w'|_0 = i \text{ mod } 2$. There are two cases to consider.

i. If $i = 0$, then $|w'|_0 = 0 \text{ mod } 2$. If $a = 0$, then $\delta(q_0, 0) = q_1$ and $|w|_0 = |w'|_0 + 1 = 1 \text{ mod } 2$. Similarly, if $a = 1$, then $\delta(q_0, 0) = q_0$ and $|w|_0 = |w'|_0 + 0 = 0 \text{ mod } 2$.

ii. If $i = 1$, then $|w'|_0 = 1 \text{ mod } 2$. If $a = 0$, then $\delta(q_1, 0) = q_0$ and $|w|_0 = |w'|_0 + 1 = 0 \text{ mod } 2$. Similarly, if $a = 1$, then $\delta(q_1, 0) = q_1$ and $|w|_0 = |w'|_0 + 0 = 0 \text{ mod } 2$. 

Thus, we have shown that $\delta(q_0, w) = q_i$ if and only if $|w|_0 = i \mod 2$. Since $q_0$ is the only accepting state of $A$, we have that $A$ recognizes a word $w$ if and only if $|w| = 0 \mod 2$.

3. Let $A$ be the DFA depicted in the state diagram that follows.

![State Diagram]

We will show that $A$ recognizes $L$ by showing that for a word $w \in \Sigma^*$, we have the following conditions:

(a) $\delta(q_0, w) = q_0$ iff $w = \varepsilon$,
(b) $\delta(q_0, w) = q_1$ iff $w = G$ or $w = w'a$ for some $w' \in \Sigma^*$ and $a \in \{A, C, T\}$,
(c) $\delta(q_0, w) = q_2$ iff $w = w'G$ for some $w' \in \Sigma^+$,
(d) $\delta(q_0, w) = q_3$ iff $w = w'GG$ for some $w' \in \Sigma^+$.

We will show this by induction on the length of $w$.

For our base case, we consider $|w| = 0$, which means $w = \varepsilon$. Since $w = \varepsilon$, we have $\delta(q_0, w) = q_0$ by definition. Thus the base case holds.

Now we consider $|w| > 0$ and assume that for all words $u \in \Sigma^*$ with $|u| < |w|$, $u$ satisfies the conditions above. Let $w = w'a$ for $a \in \Sigma$ and $w' \in \Sigma^*$. We have the following cases to consider.

(a) If $a \in \{A, C, T\}$, then for every $w' \in \Sigma^*$, we have $\delta(q_0, w') = q_1$, since for every state $q \in Q$, we have $\delta(q, a) = q_1$ by definition. Then this satisfies the condition that $\delta(q_0, w) = q_1$ if only if $w = w'a$ for $a \in \{A, C, T\}$ or $w = G$.

(b) If $a = G$, then we have the following cases to consider.

i. If $\delta(q_0, w') = q_0$, then $w' = \varepsilon$ by our inductive hypothesis. Then we have $\delta(q_0, G) = q_1$ by definition and $w = G$, satisfying the condition that $w$ is either $G$ or ends in one of $A, C, T$.

ii. If $\delta(q_0, w') = q_1$ then $w' = G$ or $w' = ub$ for some $u \in \Sigma^*$ and $b \in \{A, C, T\}$ by the induction hypothesis. We have $\delta(q_1, G) = q_2$ and $w = w'G$, as required.

iii. If $\delta(q_0, w') = q_2$ then $w' = uG$ for some $u \in \Sigma^+$ by the induction hypothesis. We have $\delta(q_2, G) = q_3$ and $w = w'G = uGG$ for some $u \in \Sigma^+$ as required.
iv. If \( \delta(q_0, w') = q_3 \), then \( w' = uGG \) for some \( u \in \Sigma^+ \) by the induction hypothesis. We have \( \delta(q_3, G) = q_3 \). Let \( v = uG \) and we have \( w = uGGG = vGG \), with \( v \in \Sigma^+ \) as required.

Since \( q_3 \) is the sole final state, \( A \) must only accept words of the form \( w = w'GG \) for \( w' \in \Sigma^+ \). Since \( w' \in \Sigma^+ \), we can write \( w' = ua \) for some \( u \in \Sigma^+ \) and \( a \in \Sigma \) and we have \( w = uaGG \). Thus, \( A \) recognizes all words ending in \( aGG \) with \( a \in \Sigma \).

4. Since \( L \) is regular, it must be recognized by a DFA \( A = (Q, \Sigma, \delta, q_0, F) \). We will show that \( L' \) is also regular by constructing a DFA \( B \) that recognizes it. Let \( B = (Q, \Sigma, \delta, q_0, F') \), where \( Q, \Sigma, \delta, q_0 \) are all as defined for \( A \) and the set of final states \( F' \) is defined by

\[
F' = \{ q \in Q \mid (\exists v \in \Sigma^*) \delta(q_0, v) \in F \}.
\]

That is, \( F' \) consists of all states \( q \in Q \) from which there exists a word that reaches a final state of \( A \). Now, we will show that \( L(B) = L' \).

First, we will show \( L(B) \subseteq L' \). Suppose \( w \in L(B) \). Let \( q = \delta(q_0, w) \in F' \). By definition of \( F' \), there exists a word \( v \in \Sigma^* \) such that \( \delta(q, v) \in F \). But this means we have \( \delta(\delta(q_0, w), v) = \delta(q_0, wv) \in F \) and therefore \( wv \in L \). Thus, \( w \in L' \) by definition of \( L' \).

Next, we will show \( L' \subseteq L(B) \). Suppose \( w \in L' \). Then there exists a word \( v \in \Sigma^* \) such that \( wv \in L \). Thus, \( wv \) is recognized by \( A \) and we have \( \delta(q_0, wv) \in F \). But this means there exists a state \( q \) such that \( \delta(q_0, w) = q \) and \( \delta(q, v) \in F \). This means we have \( q \in F' \) by definition and thus \( w \in L(B) \).

Thus, we have shown that \( L(B) = L' \).

5. Let \( A \) be the \( \varepsilon \)-NFA depicted in the state diagram that follows.

\[
\begin{array}{ccccccc}
\text{start} & \rightarrow & \langle q_0, 0 \rangle & \rightarrow & \langle q_1, 0 \rangle & \rightarrow & \langle q_3, 0 \rangle \\
& \downarrow & b & a & a,b & a,b & a,b,\varepsilon \\
\langle q_0, 1 \rangle & \rightarrow & \langle q_1, 1 \rangle & \rightarrow & \langle q_2, 1 \rangle & \rightarrow & \langle q_3, 1 \rangle \\
& a,\varepsilon & b,\varepsilon & a,b & b,\varepsilon & a,b & a,b \\
\end{array}
\]

We will show that \( A \) recognizes the language of words over \( \{a,b\} \) with an edit distance of at most 1 from the word \( baa \).

It is clear that travelling along each row of the NFA gives a path to accept the word \( baa \). We claim that travelling from row 0 to row 1 corresponds to one edit operation. In particular, for \( i = 0, 1, 2, 3 \), going from \( \langle q_i, 0 \rangle \) to \( \langle q_i, 1 \rangle \) corresponds to an insertion...
of a symbol, while going from \( (q_i, 0) \) to \( (q_{i+1}, 1) \) corresponds to a deletion operation via the \( \varepsilon \)-transition and a substitution otherwise. Only one such edit operation can be performed; otherwise, the machine crashes, as there are no outgoing transitions corresponding to edit operations from states \( (q_i, 1) \).