Turing-recognizable versus Turing-decidable

Notes for CS 360, Fall 2019

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Here is an example of the difference between Turing recognizability and Turing decidability.

Consider two languages over a 1-letter alphabet $\{a\}$, defined as follows.

$$L_{\text{sum}} = \{a^k \ : \ \exists \text{ prime numbers } p_1, p_2 \text{ such that } k = p_1 + p_2\}$$
$$L_{\text{diff}} = \{a^k \ : \ \exists \text{ prime numbers } p_1, p_2 \text{ such that } k = p_1 - p_2\}$$

Then L_{sum} is Turing-decidable, and it can be decided as follows: on input a^k, we count the number of a's to find k. Then we try every j, $0 \le j \le k/2$, and check if j and $k - j$ are both prime (using any method, such as trial division to see if there is any divisor other than the number and 1). If we find such a j, then the input is accepted; otherwise it is rejected.

On the other hand, L_{diff} is Turing-recognizable, because it can be recognized as follows: on input k, we test every integer $i \ge 2$, one after the other, to see if both i and $i + k$ are primes. If we find such an i, then we accept the input. If we never find such an i, then the Turing machine fails to halt.

Is L_{diff} also Turing-decidable? Good question! The answer is, nobody currently knows. It could be that $L_{\text{diff}} = L$, where

$$L = \{a^{p-2} \ : \ p \text{ is a prime}\} \ \cup \ (aa)^*;$$

this is known as Maillet's conjecture, and is still unresolved. If Maillet's conjecture holds, then L_{diff} would also be Turing-decidable. Of course, L_{diff} could also be Turing-decidable, even if Maillet's conjecture doesn't hold.

The big difference between the two languages L_{diff} and L_{sum} is that in the case of L_{sum}, the search space to check whether an input is accepted is bounded, **and we know what the bound is**, so we can just test all possibilities.

The search space for L_{diff}, on the other hand, is not obviously bounded. It *could* be bounded, if we knew more about prime numbers. We just don't know currently, and it is not obvious from the definition.