University of Waterloo  
CS 360 — Introduction to the Theory of Computing  
Fall 2019  
Problem Set 10

Handed out Wednesday, November 20, 2019

Due Wednesday, November 27, 2019, at 6 PM. Submit to LEARN.

This is the last problem set! Congrats for making it this far.

1. [10 marks] Recall that a square is a nonempty string that consists of one string repeated twice, like the English word murmur.

Consider the following language:

$$L = \{\langle M \rangle : M \text{ is a DTM that accepts at least one square} \}.$$

Prove that $L$ is Turing-recognizable, but not Turing-decidable.

2. [10 marks] Find two undecidable (not Turing-decidable) languages, neither of which can be reduced to the other, and prove your answer. Hints: Use the halting problem as one of your problems; use Theorem 16.3 of the course text.

3. [10 marks] Hilbert’s tenth problem is a decision problem where the instance is a multivariate polynomial with integer coefficients, such as $p(x, y, z) = 2x^3y^2 - 3xyz + 14$, and the question is: Are there values of the variables such that $p = 0$?

There are two variations of this problem: one where the variables can take on any integer value, and one where the values of the variables are restricted to be non-negative integers.

Show that these two problems each reduce to the other. That is, show that $L_1 \leq_m L_2$ and $L_2 \leq_m L_1$, where

$$L_1 := \{\langle p \rangle : p \text{ is a polynomial in } n \text{ variables and } \exists x \in \mathbb{Z}^n \text{ such that } p(x) = 0\}$$

$$L_2 := \{\langle p \rangle : p \text{ is a polynomial in } n \text{ variables and } \exists x \in \mathbb{N}^n \text{ such that } p(x) = 0\}.$$

Hint: for one direction, use Lagrange’s theorem, which states that every non-negative integer can be represented as the sum of four squares of integers.