1. [10 marks] Something is wrong with the following “proof” that the language

$$L_1 = \{ a^{n^2} : n \geq 1 \}$$

is regular. Identify the exact line that is false, and explain why it is false.

1. We will prove that $L_1$ is regular by induction.
2. Clearly $a^{n^2}$ for $n = 1$ is regular, because it is specified by the regular expression $a$. That’s the base case.
3. Now the induction step: assume $L(n) = \{ a^{i^2} : 1 \leq i < n \}$ is regular. Consider $L(n) \cup \{ a^{n^2} \}$. The left-side of the union is regular by induction. The right-side of the union is regular because it is specified by the regular expression $\overbrace{aa \cdots a}^{n^2}$. And regular languages are closed under union.
4. So $L(n)$ is regular for all $n$, and hence $L_1$ is regular.

2. [10 marks] Using the pumping lemma, prove that the language $L_2 := \{ a^i b^j : i > j \geq 0 \}$ is not regular.

3. [10 marks] Let $|x|_w$ be the number of occurrences of the word $w$ as a contiguous subword inside $x$, where overlapping occurrences are allowed. Thus, for example $|ababa|_{aba} = 2$. Consider the language

$$L_3 = \{ x \in \{0,1\}^* : |x|_{ab} = |x|_{ba} \}.$$

Is $L_3$ regular? Prove or disprove.

4. [5 marks extra credit only] Suppose $L \subseteq \Sigma^*$ is a regular language. Must

$$K(L) := \{ x \in \Sigma^* : xx^R \in L \}$$

also be regular? Prove your answer.