1. [10 marks] Something is wrong with the following “proof” that the language

\[ L_1 = \{a^{n^2} : n \geq 1\} \]

is regular. Identify the exact line that is false, and explain why it is false.

1. We will prove that \( L_1 \) is regular by induction.

2. Clearly \( a^{n^2} \) for \( n = 1 \) is regular, because it is specified by the regular expression \( a \). That’s the base case.

3. Now the induction step: assume \( L(n) = \{a^{i^2} : 1 \leq i < n\} \) is regular. Consider \( L(n) \cup \{a^{n^2}\} \). The left-side of the union is regular by induction. The right-side of the union is regular because it is specified by the regular expression \( aa \cdots a \). And regular languages are closed under union.

4. So \( L(n) \) is regular for all \( n \), and hence \( L_1 \) is regular.

2. [10 marks] Using the pumping lemma, prove that the language \( L_2 := \{a^ib^j : i > j \geq 0\} \) is not regular.

3. [10 marks] Let \( |x|_w \) be the number of occurrences of the word \( w \) as a contiguous subword inside \( x \), where overlapping occurrences are allowed. Thus, for example \( |ababa|_ababa = 2 \). Consider the language

\[ L_3 = \{x \in \{a,b\}^* : |x|_{ab} = |x|_{ba}\} \]

Is \( L_3 \) regular? Prove or disprove.

4. [5 marks extra credit only] Suppose \( L \subseteq \Sigma^* \) is a regular language. Must \( K(L) := \{x \in \Sigma^* : xx^R \in L\} \) also be regular? Prove your answer.