CS452 : TIMELESS

REVERSE ENGINEERING ACCELERATION/DECELERATION

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1. INTRODUCTION

It is common, when writing application programs that deal with the real world, to interact with external systems the properties of which are unknown. It is then necessary to discover the properties of the external system as part of creating a calibration. The acceleration/deceleration (AD) characteristics of trains is an instructive example: this document describes a possible process in some detail.

2. ASSUMPTIONS

We start with a few plausible assumptions.

1. The AD characteristics are not the dynamics of a high-speed electric motor powering a relatively light load, but simulate the characteristics of a high torque, low power electric motor powering a very heavy load.
2. The AD characteristics were programmed into a microcontroller for hobbyists. Thus, they are likely to be simple, but conform to the qualitative characteristics of real train locomotives.
3. The programmer is likely to re-use code as much as possible, not out of good programming practice, but out of pure sloth.
4. We will 'solve' the generic problem of a train travelling at velocity v_0 at time t_0, which is to have velocity v_1 at time $t_{1.}$. The acceleration $a(t)$ should be zero at the beginning and end of the A/D interval. This gives us four conditions to satisfy.
5. We hope that the same solution will work anywhere on the track and at any time. Thus, without loss of generality we can set $t_0 = 0$ and $x(0) = 0$, the latter amounting to a fifth condition.

Based on these assumptions we guess that the AD function is a fourth order polynomial. Why?

1. There are five free parameters in a fourth order polynomial, which will be determined by the five conditions.
2. This is probably the simplest function with five free parameters.
3. I think I remember students from earlier terms mentioning that they calibrated AD with velocity as a third order polynomial.
4. We should notice, by eye, discontinuities in location, velocity and acceleration, so these three functions must be continuous.

3. Solution

The general fourth order polynomial is

$$x(t) = \frac{1}{24}A\left(\frac{t-t_0}{t_1-t_0}\right)^4 + \frac{1}{6}B\left(\frac{t-t_0}{t_1-t_0}\right)^3 + \frac{1}{2}C\left(\frac{t-t_0}{t_1-t_0}\right)^2 + D\left(\frac{t-t_0}{t_1-t_0}\right) + E .$$

Because $x(0) = x_0$, $E = x_0$. We then have

$$\begin{aligned} v(t) &= \frac{1}{6}A\left(\frac{t-t_0}{t_1-t_0}\right)^3 + \frac{1}{2}B\left(\frac{t-t_0}{t_1-t_0}\right)^2 + C\left(\frac{t-t_0}{t_1-t_0}\right) + D \\ a(t) &= \frac{1}{2}A\left(\frac{t-t_0}{t_1-t_0}\right)^2 + B\left(\frac{t-t_0}{t_1-t_0}\right) + C \\ j(t) &= A\left(\frac{t-t_0}{t_1-t_0}\right) + B \end{aligned} .$$

Because $a(0) = 0$, $C = 0$. Because $v(0) = v_0$, $D = v_0$. To find the remaining parameter values use the end point conditions, $a(t_1) = 0$, and $v(t_1) = v_1$. A little high school algebra gives

$$\begin{aligned} A &= -\frac{12}{(t_1-t_0)^3}(v_1-v_0) \\ B &= \frac{6}{(t_1-t_0)^2}(v_1-v_0) \end{aligned} ,$$

so that

$$\begin{aligned} x(t) &= -\frac{1}{2}(t_1-t_0)(v_1-v_0)u^4 + (t_1-t_0)(v_1-v_0)u^3 + (t_1-t_0)v_0u + x_0 \\ v(t) &= -2(v_1-v_0)u^3 + 3(v_1-v_0)u^2 + v_0 \\ a(t) &= -\frac{6(v_1-v_0)u(u-1)}{(t_1-t_0)} \\ j(t) &= -\frac{6}{t_1^2}(v_1-v_0)u + \frac{6}{t_1^2}u \end{aligned} ,$$

where we have defined $u = \dfrac{t-t_0}{t_1-t_0}$ to simplify the equations that follow.

v_0 and v_1 we know from our velocity calibration. How do we find t_1?

3.A. STOPPING DISTANCE METHOD (CALLED THE 'EASY' METHOD IN CLASS)
There is a special case, stopping. We can easily measure the stopping distance, which is the case when $v_1 = 0$. Then the stopping distance is

$$\begin{aligned} x(t_1) &= \frac{1}{2}(t_1 - t_0)v_0 = d \\ t_1 - t_0 &= \frac{2d}{v_0} \end{aligned}.$$

It should be easy to verify this relationship. Measure the stopping time using a stop watch, then check it against the stopping distances and velocites you measured for different speeds. Even without knowing the stopping time, you know the velocity and position of the train while it is stopping:

$$\begin{aligned} x(t) &= d\left(\frac{v_0 t}{2d}\right)^4 - 2d\left(\frac{v_0 t}{2d}\right)^3 + 2d\left(\frac{v_0 t}{2d}\right) = v_0 t\left(\left(\frac{v_0 t}{2d}\right)^3 - 2\left(\frac{v_0 t}{2d}\right)^2 + 2\right) \\ v(t) &= 2v_0\left(\frac{v_0 t}{2d}\right)^3 - 3v_0\left(\frac{v_0 t}{2d}\right)^2 + v_0 = v_0\left(2\left(\frac{v_0 t}{2d}\right)^3 - 3\left(\frac{v_0 t}{2d}\right)^2 + 1\right) \end{aligned}.$$

This covers most of the cases that will occur in your project.

3.B. TRAVEL TIME METHOD (CALLED THE 'DIFFICULT' METHOD IN CLASS)
Choose two sensors that are far enough apart, but not too far. Do the following.

1. Measure the time you pass the first sensor, t_{-1}.
2. Measure the time you give the command to change speed, t_0.
3. The time at which the change of velocity is complete, t_1, is unknown.
4. Measure the time you pass the second sensor, t_2, which is chosen to be greater than $.t_1$

The distance the train travels between t_{-1} and t_2 is the distance between the two sensors. Call it d_T. It is broken into three parts, the distance before the command, $d_{-1} = v_0(t_0 - t_{-1})$, the distance after the change of velocity is complete, $d_1 = v_1(t_2 - t_1)$, and the distance travelled during the change of velocity, d, where

$$d = \frac{1}{2}(t_1 - t_0)(v_1 + v_0) = d_T - v_0(t_0 - t_{-1}) - v_1(t_2 - t_1).$$

Thus,

$$\begin{aligned}
\frac{1}{2}(t_1 - t_0)(v_1 + v_0) &= d_{\mathrm{T}} - v_0(t_0 - t_{-1}) - v_1(t_2 - t_1) \\
\frac{1}{2}t_1(v_1 + v_0) - t_1 v_1 &= d_{\mathrm{T}} - v_0(t_0 - t_{-1}) - v_1 t_2 + \frac{1}{2}t_0(v_1 + v_0) \\
\frac{1}{2}t_1(-v_1 + v_0) &= d_{\mathrm{T}} + v_0 t_{-1} - v_1 t_2 + \frac{1}{2}t_0(v_1 - v_0) \\
t_1 &= -t_0 + \frac{2(t_2 v_1 - t_{-1} v_0 - d_T)}{v_1 - v_0}
\end{aligned}.$$

Checks on this result, which does not amell right to me.

Translation in time. Translate by Δt : the result should change by the same amount.

$$\begin{aligned}
& -(t_0 + \Delta t) + \frac{2((t_2 + \Delta t)v_1 - (t_{-1} + \Delta t)v_0 - d_T)}{v_1 - v_0} \\
&= \left(-t_0 + \frac{2(t_2 v_1 + t_{-1} v_0 - d_T)}{v_1 - v_0}\right) + \left(-\Delta t + \frac{2(\Delta t v_1 - \Delta t v_0)}{v_1 - v_0}\right), \\
&= t_1 + \Delta t\left(-1 + \frac{2(v_1 - v_0)}{v_1 - v_0}\right) \\
&= t_1 + \Delta t
\end{aligned}$$

as desired.

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3.c. Point of Inflection Method

Because the deceleration goes smoothly to zero there should be nothing to see at t_1 [1]. Perhaps, however, we can spot the half-way point. There acceleration changes to deceleration, so you may see slack motion in the gears give the train a little wobble. If using a video camera, observe that there is a symmetry about the mid-way point: the deceleration profile is identical to the acceleration profile with the sign reversed. Finding the mid-point of a symmetric function can be done with more precision than finding where its third derivative goes to zero. Here are a couple of other interesting facts about this function, which can be deduced from symmetry alone:

$$\begin{aligned}
x(\frac{1}{2}t_1) &= \frac{1}{4}(v_1 + v_0)t_1 \\
x(t_1) &= \frac{1}{2}(v_1 + v_0)t_1
\end{aligned}.$$

1. Even looking carefully with a video camera this spot will be very hard to locate

Using them it is possible to look for locations where things happen rather than times at which they happen.[2]

4. Summary

This document is not presented as a set of instructions for calibrating A/D in a train. It is presented as a concrete example of the process we follow when reverse engineering.

1. Make reasonable assumptions.
2. Deduce a plausible parametrized model from the assumptions.
3. Solve the parametrized model in terms of measurable quantities.
4. Measure the quantities.
5. Check that the model is robust.

2. What's good about a video camera is that it measures time, in terms of frames, and distance, in terms of pixels, at the same time.