Problem:
Show that a string $w$ is bordered iff some nonempty proper prefix of $w$ is a conjugate of a suffix of $w$.

Solution:
For the forward direction, suppose $w$ is bordered. That is, $w = xu = vx$ where $x \in \Sigma^+, u, v \in \Sigma^*$. Then $x$ is a nonempty proper prefix of $w$ and is equal to a suffix of $w$. $x$ is trivially a conjugate of itself, hence it is also a conjugate of a suffix of $w$.

For the reverse direction, suppose some nonempty proper prefix of $w$ is a conjugate of a suffix of $w$. That is, $w = uvx = yvu$ where $u, v, x, y \in \Sigma^*$ and at least one of $u$ and $v$ is nonempty. We see that if $u$ is nonempty, then $u$ is a border of $w$. Otherwise, $w = vx = yv$ and $v$ must be nonempty, so $v$ is a border of $w$. Therefore $w$ is bordered.