Define \( \min(L) = \{ x \in L : \text{no proper prefix of } x \text{ is in } L \} \)

Find an expression for \( \min(L) \) and conclude that if \( L \) is regular, so is \( \min(L) \).

**SOLUTION** : Show that \( (L \cap \overline{L \cdot \Sigma^+}) = \min(L) \) using double inclusion.

First, show \( (L \cap \overline{L \cdot \Sigma^+}) \subseteq \min(L) \). \( (L \cdot \Sigma^+) \) is the set of all strings in \( \Sigma^* \) with a proper prefix in \( L \). Therefore, \( \overline{L \cdot \Sigma^+} \) is the language of strings in \( \Sigma^* \) that do not have a proper prefix \( x \in L \). Then by intersecting this set with \( L \), we ensure that \( x \in L \) and arrive at the definition of \( \min \).

Next, show \( \min(L) \subseteq (L \cap \overline{L \cdot \Sigma^+}) \). Then \( x \in \min(L) \Rightarrow x \in L \). Furthermore, no proper prefix of \( x \) is in \( L \) implies that \( x \notin (L \cdot \Sigma^+) \). Altogether,

\[
\begin{align*}
x \in \min(L) & \Rightarrow x \in L \\
x \in \min(L) & \Rightarrow x \notin (L \cdot \Sigma^+) \\
x \notin (L \cdot \Sigma^+) & \Rightarrow x \in \overline{L \cdot \Sigma^+} \\
x \in L \text{ and } x \in \overline{L \cdot \Sigma^+} & \Rightarrow x \in (L \cap \overline{L \cdot \Sigma^+})
\end{align*}
\]

Finally, if \( L \) is regular, then \( (L \cap \overline{L \cdot \Sigma^+}) \) is also regular since each operation in the expression is closed under regular languages. Since \( (L \cap \overline{L \cdot \Sigma^+}) = \min(L) \), then if \( L \) is regular, so is \( \min(L) \).