Problem 18

Is the following language regular? \( A = \{ xwx^R : x, w \in \{0,1\}^+ \} \)

Solution

The language \( A \) is regular and can be represented by the regular expression

\[
R = 0\Sigma\Sigma^*0 \mid 1\Sigma\Sigma^*1
\]

where \( \Sigma = \{0,1\} \). \( A \) seems deceptively non-regular at first glance due to its resemblance to the language of binary palindromes (which can be shown non-regular by the pumping lemma for the regular languages).

1. \( A \subseteq L(R) \) : If \( v = xwx^R \in A \), then \( v \) starts and ends with the same letter \( a \in \{0,1\} \). In particular, \( x \) starts with \( a \) and \( x^R \) ends with \( a \). So we can write \( x = ax' \) and \( v = a(x'wx'^R) \). Here, \( x'wx'^R \in \Sigma\Sigma^* \) because \( |w| \geq 1 \). This shows that \( v \) can be recognized by \( R \), and \( v \in L(R) \).

2. \( L(R) \subseteq A \) : Let \( u = awa \in L(R) \) where \( a = 0,1 \) and \( w \in \Sigma^+ \). If we let \( x = a \) and \( x^R \in \Sigma^+ \), then \( u = xwx^R \in A \).

This shows \( A = L(R) \). Since \( A \) can be described by a regular expression, \( A \) is regular. \( \Box \)