Problem 23. Let $L_1, L_2$ be regular languages with $L_2 - L_1$ infinite. Show there exists a regular language $L$ with $L_1 \subseteq L \subseteq L_2$ with both $L - L_1$ and $L_2 - L$ infinite. (Here $A - B$ is set difference, defined to be those strings in $A$ but not in $B$.)

Proof. Assume that $L_1 \subseteq L_2$ (otherwise we can’t find a language $L$ such that $L_1 \subseteq L \subseteq L_2$).

$L_2 - L_1$ is regular by the closure of regular languages under set difference. Since $L_2 - L_1$ is regular, it has some pumping length $p$. $L_2 - L_1$ is infinite so it must contain some word $z$ where $|z| \geq p$.

We pump $z$: By the pumping lemma for regular languages, there must be some non-empty string $w$ and strings $x, y$ such that $z = xwy$ and $xw^iy \in L_2 - L_1$ for all $i \in \mathbb{N}$.

Let $L := L_1 \cup \{xw^iy : i \in \mathbb{N}\}$. We can write a regular expression for $\{xw^iy : i \in \mathbb{N}\}$ as $x(ww)^*y$, so $L$ is regular by the closure of regular languages under union.

Clearly $L_1 \subseteq L$. Since $L_2 = L_1 \cup (L_2 - L_1)$ and $\{xw^iy : i \in \mathbb{N}\} \subseteq L_2 - L_1, L \subseteq L_2$. $L - L_1 = \{xw^iy : i \in \mathbb{N}\}$ so $L - L_1$ is infinite.

$L_2 - L \supseteq \{xw^{2i+1}y : i \in \mathbb{N}\}$ so $L_2 - L$ is infinite.

Thus $L$ is regular, $L_1 \subseteq L \subseteq L_2$ and both $L - L_1$ and $L_2 - L$ are infinite languages, as required. \qed