Problem 24

Let $L_1, L_2$ be regular languages with $L_2 - L_1$ infinite. Show there exists a regular language $L$ with $L_1 \subseteq L \subseteq L_2$ with both $L_2 - L$ and $L - L_1$ infinite. (Here $A - B$ is set difference, defined to be those strings in $A$ but not in $B$.)

Let $Q = L_2 - L_1$. As $L_2$ and $L_1$ are regular languages, $Q$ is regular, and from the question, it is infinite. By the pumping lemma there exists a pumping length $n \geq 1$ such that if we pick some $w$ of length $\geq n$, then we can write: $w = xyz$ where $x, y, z \in \Sigma^*$ and:

- $|xy| \leq n$
- $|y| \geq 1$
- $xy^iz \in Q$ for every $i \in \mathbb{N}$

Define $Q' = \{xy^iz : i \in \mathbb{N}, i \text{ is odd}\}$. So $Q'$ is an infinite regular language such that $Q' \subset Q = L_2 - L_1$

Let $L = Q' \cup L_1$

Now let’s show this $L$ satisfies all our requirements.

1. $L_1 \subseteq Q' \cup L_1 = L$
2. $L = Q' \cup L_1 \subseteq Q \cup L_1 = (L_2 - L_1) \cup L_1 \subseteq L_2$
3. $L_2 - L = L_2 - (Q' \cup L_1) = Q - Q'$ which is infinite
4. $L - L_1 = (Q' \cup L_1) - L_1 = Q'$ which is infinite