1. Define the following transformation on languages, similar to log($L$) in the text, p. 76:

$$\sqrt{L} = \{ x \in \Sigma^* : \exists y \text{ such that } |y| = |x|^2 \text{ and } xy \in L \}$$

Show that if $L$ is regular, then so is $\sqrt{L}$. Hint, modify the construction we used for log($L$).

Given a DFA $A = (Q, \Sigma, \delta, q_0, F)$ for $L$, let $M_a$ be the incidence matrix corresponding to the input symbol $a$. Let $M$ be the Boolean ‘or’ of all the matrices $M_a$. Then $M$ has the property that there is an entry in row $i$ and column $j$ if and only if there is a transition in $A$ on some symbol from $q_i$ to $q_j$.

We now make a DFA $A' = (Q', \Sigma, \delta', q'_0, F')$ for log($L$). Here

$Q' = \{ [B,C,D] : B, C, D \text{ are } n \times n \text{ Boolean matrices} \}$

where $n$ is the number of states in $Q$. The basic idea is that if on input $x$ we arrive at the state $[B,C,D]$, then $B = M_x$, $C = M^{|x|^2}$, and $D = M^{|x|}$. To enforce this, we set $q'_0 = [I_{n \times n}, M, M]$ and define $\delta'([B,C,D], a) = [BM_a, CD^2M, DM]$. We also set

$F' = \{ [B,C,D] : BC \text{ has a 1 in row 0 and column } j \text{ such that } q_j \in F \}$

Then $x$ is accepted by $A'$ if and only if $M_x M^{|x|^2}$ has a 1 in row 0 and a column corresponding to a final state, which occurs if and only if there exists $y$, $|y| = |x|^2$, such that $xy \in L(M)$. 