Q29. Let $M$ be an NFA. Show that the set of all strings in $L(M)$ having exactly one accepting path is a regular language.

**SOLUTION**

Set $M = (Q, \Sigma, \delta, q_0, F)$ and let $N_a$ be the incidence matrix corresponding to the transitions on symbol $a \in \Sigma$.

Define a new operation $\cdot_p$ on matrices which behaves like ordinary matrix multiplication, except if an entry in the resulting matrix would be $\geq 2$, set it to 2. We make this limitation so that we have a finite number of states in the new DFA that we define.

Define a new DFA $M' = (Q', \Sigma, \delta', q'_0, F')$ which will recognize the strings of $L(M)$ that have exactly one accepting path. Set

\[ Q' = \{ A : A \text{ is an } n \times n \text{ matrix of elements } 0, 1, 2 \} \]  
\[ \delta'(A, a) = A \cdot_p N_a \]  
\[ q'_0 = I \]  
\[ F' = \{ A : u \cdot_p A \cdot_p v \} \]

where $n$ is the number of states in $M$ and $I$ is the $n \times n$ identity matrix. Additionally, $u$ is the $n$ length vector $[1, 0, ..., 0]$ and $v$ is the $n$ length vector where the $i$-th entry is 1 if $q_i \in F$ and 0 otherwise.

Each matrix $A$ is some $N_x$ for $x \in \Sigma^*$ and represents the number of paths from $q_i$ to $q_j$ upon reading input $x$. The representation of 0 and 1 path is unchanged, and if the number of paths is $\geq 2$, then the corresponding entry in $A$ will be 2. Then we limit $F'$ to those matrices which represent a single path from a start state to an accepting state.

Then we need to ensure that $(AB)[i, j] = 1$ under ordinary matrix multiplication if and only if $(A \cdot_p B)[i, j] = 1$. The forward direction is easy, since the definition of $\cdot_p$ does not modify a result of 1. For the backward direction assume that $(A \cdot_p B)[i, j] = 1$ and assume towards contradiction that $(AB)[i, j] \neq 1$. Then in case 1, $(AB)[i, j] = 0$ and $(A \cdot_p B)[i, j]$ should be 0, a contradiction. In case 2, $(AB)[i, j] \geq 2$ so $(A \cdot_p B)[i, j]$ should be 2, a contradiction.

Then we have that an input string $x \in L(M')$ iff $M$ has a single accepting path for $x$, and so the language in question is regular.