Lemma \[ \mu(\overline{x}) = \overline{\mu(x)} \]

Proof: induction on \(|x|\)

Base case: \( \mu(\overline{o}) = \mu(1) = 0 \overline{1} = \overline{\mu(o)} \)
  Similar for \(1\)

Induction: let \( x = ax' \)
  \[ \mu(\overline{ax'}) = \mu(\overline{a}) \mu(\overline{x'}) \]
  by definition of morphism
  \[ = \frac{\mu(a)}{\overline{\mu(x)}} \]
  by base case
  and inductive hypothesis

Claim \[ \mu^n(a) = \mu^{n-1}(a) \overline{\mu^{n-1}(a)}, \quad n \geq 2 \]

Proof: induction on \(n\)

Base case: \( \mu^2(o) = 0110 = 0(1)\overline{0(1)} \)
  Similar for \(1\)

Induction: \[ \mu^n(a) = \mu(\mu^{n-1}(a)) \]
  \[ = \mu(\mu^{n-2}(a) \overline{\mu^{n-2}(a)}) \]
  by inductive hypothesis
  \[ = \mu(\mu^{n-2}(a)) \mu(\overline{\mu^{n-2}(a)}) \]
  by definition of morphism
  \[ = \mu(\mu^{n-2}(a)) \mu(\mu^{n-2}(\overline{a})) \]
  by Lemma
  \[ = \mu^{n-1}(a) \overline{\mu^{n-1}(a)} \]
  by Lemma

Suppose \( x \) is a subword of \( \overline{x} \) and
\( x \) occurs in finitely many positions of \( \omega \).

Suppose the last occurrence of \( x \) in \( \mu^n(o) \).

Then \( \mu^{n+2}(o) = \mu^{n+1}(o) \mu^{n+1}(o) \cong \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \).

\( \mu^{n+2}(o) = \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \).

And \( x \) occurs twice in \( \mu^{n+2}(o) \), a contradiction.

So, \( \mu \) is recurrent.

Simpler proof:

\[ \mu^{n+2}(o) = \mu^n(\mu^n(o)) = \mu^n(0110) = \mu^n(o) \mu^n(1) \mu^n(1) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \mu^n(o) \]

And \( \mu^{n+2} \) has 2 occurrences of \( x \), a contradiction.