CS462 Group Problem-Solving Session 3 Question 3

**Claim:** If $L_1, L_2$ are regular languages, then the perfect shuffle $L_1 \Pi L_2$ is a regular language.

**Proof:** Suppose $L_1, L_2 \subseteq \Sigma^*$. We’ll consider the set of pairs of symbols in $\Sigma$, denoted $\Sigma^2$. Define the following morphisms:

$s : (\Sigma^2)^* \rightarrow \Sigma^*$ such that for every pair $(a, b) \in \Sigma^2$ (with $a, b \in \Sigma$), $s((a, b)) = ab$. Intuitively, $s$ converts each pair into a string with the same contents; it splits the pair.

$l : (\Sigma^2)^* \rightarrow \Sigma^*$ such that for every pair $(a, b) \in \Sigma^2$, $l((a, b)) = a$. Intuitively, $l$ takes the left element of each pair.

$r : (\Sigma^2)^* \rightarrow \Sigma^*$ such that for every pair $(a, b) \in \Sigma^2$, $r((a, b)) = b$. Intuitively, $r$ takes the right element of each pair.

Now consider $l^{-1} : \Sigma^* \rightarrow 2^{(\Sigma^2)^*}$. For a string $w = a_1a_2...a_n$, $l^{-1}(w)$ is the set of all strings of pairs $(a_1, b_1)(a_2, b_2)...(a_n, b_n)$, with $b_i \in \Sigma, 1 \leq i \leq n$. Similarly, for a string $x = b_1b_2...b_n$, $r^{-1}(x)$ is the set of all strings of pairs $(a_1, b_1)(a_2, b_2)...(a_n, b_n)$, with $a_i \in \Sigma, 1 \leq i \leq n$. That is, the inverse morphisms specify one half of every pair in the output string, and leave the other half arbitrary.

Consider $l^{-1}(w) \cap r^{-1}(x)$. Clearly, if $|w| \neq |x|$, the intersection is empty. Otherwise, it contains the strings of pairs $(a_1, b_1)(a_2, b_2)...(a_n, b_n)$ where $w = a_1a_2...a_n$ and $x = b_1b_2...b_n$. The intersection thus specifies exactly one string of pairs. All that remains is to apply $s$: $s(l^{-1}(w) \cap r^{-1}(x)) = \{a_1b_1a_2b_2...a_nb_n\}$. That is, $s$ splits the pairs to convert back to our original alphabet. Observe that this matches the definition of perfect shuffling; $s(l^{-1}(w) \cap r^{-1}(x)) = \{w\Pi x\}$.

We can apply all of these operations to the languages $L_1, L_2$ to find that $s(l^{-1}(L_1) \cap r^{-1}(L_2))$ is $\{w\Pi x \mid w \in L_1, x \in L_2\} = L_1 \Pi L_2$. Since morphism, inverse morphism, and intersection all preserve regularity, if $L_1$ and $L_2$ are regular, then $L_1 \Pi L_2$ is regular. \qed