Let $L \subseteq \Delta^*$ be regular. Let $s : \Sigma^* \rightarrow 2^{\Delta^*}$ be a substitution by regular languages.

**Claim:** $s^{-1}(L) = \{ x : s(x) \subseteq L \}$ is regular.

**Proof.** $L$ is regular, so there exists a DFA $M = (Q, \Delta, \delta, q_0, F)$ accepting it. We construct a DFA $M' = (Q', \Sigma, \delta', q'_0, F')$ to accept $s^{-1}(L)$. We first illustrate how $M'$ will operate. Suppose $M'$ has read some prefix $x'$ of an input string $x$. $M'$ will maintain, as state information, the set of states in $Q$ reachable in $M$ by some string in $s(x')$. Then, when the entire string $x$ is read, $M'$ accepts if the set of states in $Q$ (reachable by some string in $s(x)$ starting from $q_0$) is a subset of $F$.

That is,

- $Q' := 2^Q$
- $\Sigma$ is the finite alphabet whose Kleene closure is the domain of our substitution $s$.
- $\delta'$ is defined as follows:
\[ \delta' : 2^Q \times \Sigma \rightarrow 2^Q \]
\[ (P, \sigma) \mapsto \{ q \in Q : \exists q' \in P, w \in s(\sigma) : \delta(q', w) = q \} \]
- $q'_0 := \{ q_0 \}$
- $F' := 2^F$

**Invariant:**

\[ \delta'(q_0, x') = \{ q \in Q : \exists w \in s(x') : \delta(q_0, w) = q \} \]

We prove this by induction on $|x'|$. In the base case that $|x'| = 0$, we have not read any symbols. So, $M'$ will be in the state $q'_0 = \{ q_0 \}$. By definition of substitutions, $\epsilon \in s(\epsilon)$ (since $s(\epsilon) = \{ \epsilon \}$). We also have $\delta(q_0, \epsilon) = q_0$. So, our invariant holds in the base case.

Suppose the invariant holds for $|x'| = n - 1$. Consider the case in which $|x'| = n$. Write $x'$ as $x''\sigma$. By the inductive hypothesis, $M'$ will be in the following state after $x''$ is read:

\[ \delta'(q_0, x'') = \{ q \in Q : \exists w \in s(x'') : \delta(q_0, w) = q \} \]

So, after also reading $\sigma$, $M'$ will be in the following state:

\[ \delta'(q_0, x''), \sigma = \{ q \in Q : \exists q' \in \delta'(q_0, x'') \exists w \in s(\sigma) : \delta(q', w) = q \} \]

By definition of $\delta'$

Rewriting $q' \in \delta'(q_0, x'')$

Reorder clauses

Combine

Drop non-binding quantifier

Property of substitution

But $\delta'(q_0, x''), \sigma = \delta'(q_0, x''\sigma) = \delta'(q_0, x')$, so our invariant holds.

Now we use our invariant to show that $M'$ accepts $s^{-1}(L)$:

\[ M' \text{ accepts } x \iff \delta'(q_0, x) \in F' \]

\[ \iff \{ q \in Q : \exists w \in s(x) : \delta(q_0, w) = q \} \subseteq F' \]

\[ \iff \forall w \in s(x) : \delta(q_0, w) \in F \]

\[ \iff \forall w \in s(x) : w \in L \]

\[ \iff s(x) \subseteq L \]

\[ \iff x \in s^{-1}(L) \]

\[ \square \]