1. Show that for every infinite string \( w \) there must be some letter \( a \) and some finite string \( x \) such that \( axa \) appears infinitely often as a subword of \( w \). Furthermore, such an \( x \) exists with \( |x| \leq |\Sigma| - 1 \) where \( \Sigma \) is the alphabet.

**Solution:**

We can divide the infinite word \( w \) into subwords of size \( k + 1 \), i.e. we write

\[
w = w_1 w_2 \ldots
\]

where \( |w_i| = k + 1 \) for all \( i \geq 1 \).

The sequence \( w_1, w_2, \ldots \) is infinite since otherwise \( w \) consists of finitely many words of finite length and so it will have finite length, contradicting its choice.

Let \( S \) be the set of words of length \( k + 1 \) over the alphabet of \( \Sigma \), and we have \( |S| = |\Sigma|^{k+1} = k^{k+1} \). Note that for every \( i \geq 1 \) we have \( |w_i| = k + 1 \) and the fact that \( w_i \) is a string over the alphabet \( \Sigma \) implies \( w_i \in S \). Hence, by the (infinite) Pigeonhole Principle there must be a word \( v \in S \) that appears infinitely many times in the sequence \( (w_1, w_2, \ldots) \). Therefore, it appears infinitely many times in \( w \).

Then, since \( |v| = k + 1 \), by the Pigeonhole Principle, there is a letter \( a \in \Sigma \) that appears at least twice in \( v \). Therefore, we can write \( v = v'(axa)v'' \) for \( x, v', v'' \in \Sigma^* \) and so \( axa \) is a subword of \( v \). Since \( v \) appears infinitely many times in \( w \), we have that \( axa \) appears infinitely many times in \( w \). We also have

\[
2 + |x| = |axa| \leq |v| = k + 1 = |\Sigma| + 1
\]

which implies \( |x| \leq |\Sigma| - 1 \). This completes the proof.