Question 7

Proof. Define our string to be $S = \prod a_n$ where $a_n = 10^n$. It’s quite easy to see that our string has no infinite power. Since if $S$ contains an infinite power of $k$ with length $n$, it must contain at least one 1. Later we can look at $a_{2n}$ which has $2n$ 0s, there is no way you can represent fit $k$ as a substring within the $2n$ zeroes. Then we claim $S$ has a square at every point. Pick any letter $x$. $X$ is in some $a_n$, and there are 3 cases.

Case 1: $x$ is a 1, then take $a_n$ and the first $n+1$ letters of $a_{n+1}$, and we have a square. This is $(10^n)^2$.

Case 2: $x$ is a 0, followed by 0. 00 is our square.

Case 3: $x$ is a 0 followed by 1. then take the following $2n+4$ letters. This is $(010^n)^2$, another square. Thus we show $x$ has a square starting at it in all 3 cases.