Recall that a word $x$ is a palindrome if $x = x^R$, where $x^R$ denotes the reversal of $x$.

For a word $x \in \{0,1\}^*$, let $\overline{x}$ denote the word obtained by changing each 0 to 1 and vice versa. Call a word $x$ an antipalindrome if $\overline{x} = x^R$. Thus, for example, 001101 is an antipalindrome.

Recall that $\Pi$ denotes the “perfect shuffle”, so that, for example, clip$\Pi$aloe = calliope.

1. Show that $x$ is an even-length palindrome if and only if there exists a string $y$ such that $x = y \Pi y^R$.

2. Call a language $L$ commutative if for all $x, y \in L$ we have $xy = yx$. Show that $L$ is commutative if and only if there exists a word $w$ such that $L \subseteq w^*$.

3. Show that for every infinite string $w$ there must be some letter $a$ and some finite string $x$ such that $axa$ appears infinitely often as a subword of $w$. Furthermore such an $x$ exists with $|x| \leq |\Sigma| - 1$, where $\Sigma$ is the alphabet.

4. Can you construct an aperiodic infinite binary word in which there is a square beginning at every position? Here “aperiodic” means “not ultimately periodic”.

5. Prove that a string $x$ is an antipalindrome if and only if there exists $y$ such that $x = y \overline{y}^R$.

6. Prove or disprove: for all integers $m, n \geq 1$, $x^m$ is an antipalindrome if and only if $x^n$ is an antipalindrome.

7. When is the concatenation of two antipalindromes an antipalindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.

8. When is the concatenation of two antipalindromes a palindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.