Here’s how these problem-solving sessions work.

Start by working on the problem indicated by the number on the front of this sheet. If you find a solution, let me know and sketch it for me. If it’s correct, you then present it on the board to the class. Next, you write up your solution within one week, in a format I can post for the rest of the class, and send it to me. If you do all that, you fulfill your 10% course mark allocated to these sessions.

If you quickly succeed in solving a problem (some of them are very easy!), try one of the extra problems at the end.

Recall that a word $x$ is a palindrome if $x = x^R$, where $x^R$ denotes the reversal of $x$.

For a word $x \in \{0, 1\}^*$, let $\overline{x}$ denote the word obtained by changing each 0 to 1 and vice versa. Call a binary word $x$ an antipalindrome if $\overline{x} = x^R$. Thus, for example, 001011 is an antipalindrome.

Recall that $\Pi$ denotes the “perfect shuffle”, so that, for example, clip$\Pi$aloe = calliope.

1. Show that a string $w$ is bordered iff some nonempty proper prefix of $w$ is a conjugate of a suffix of $w$.

2. Show that the number of bordered words of length $n$ is the same as the number of words of length $n$ that begin with a nonempty even-length palindrome. Hint: find a bijection.

3. Is the number of bordered words of length $n$ the same as the number of words of length $n$ that begin with a nonempty square? Prove or disprove.

4. Show that $x$ is an even-length palindrome if and only if there exists a string $y$ such that $x = y \Pi y^R$. 

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5. Call a language $L$ commutative if for all $x, y \in L$ we have $xy = yx$. Show that $L$ is commutative if and only if there exists a word $w$ such that $L \subseteq w^*$.

6. Show that for every infinite string $w$ there must be some letter $a$ and some finite string $x$ such that $axa$ appears infinitely often as a subword of $w$. Furthermore such an $x$ exists with $|x| \leq |\Sigma| - 1$, where $\Sigma$ is the alphabet.

7. Can you construct an aperiodic infinite binary word in which there is a square beginning at every position? Here “aperiodic” means “not ultimately periodic”.

8. Prove that a string $x$ is an antipalindrome if and only if there exists $y$ such that $x = yy^R$.

9. Prove or disprove: for all integers $m, n \geq 1$, $x^m$ is an antipalindrome if and only if $x^n$ is an antipalindrome.

Additional Problems — if you already solved your group problem try these

10. Can you construct an aperiodic infinite word in which there are powers of arbitrarily large exponent beginning at every position? Hint: construct it iteratively.

11. When is the concatenation of two antipalindromes an antipalindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.

12. When is the concatenation of two antipalindromes a palindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.