Here’s how these problem-solving sessions work.

Start by working on the problem indicated by the number on the front of this sheet. If you find a solution, let me know and sketch it for me. If it’s correct, you then present it on the board to the class. Next, you write up your solution within one week, in a format I can post for the rest of the class, and send it to me. If you do all that, you fulfill your 10% course mark allocated to these sessions.

If you quickly succeed in solving a problem (some of them are very easy!), try one of the extra problems at the end.

Recall that a word \( x \) is a palindrome if \( x = x^R \), where \( x^R \) denotes the reversal of \( x \).

For a word \( x \in \{0, 1\}^* \), let \( \overline{x} \) denote the word obtained by changing each 0 to 1 and vice versa. Call a binary word \( x \) an antipalindrome if \( \overline{x} = x^R \). Thus, for example, \( 001011 \) is an antipalindrome.

Recall that \( \text{III} \) denotes the “perfect shuffle”, so that, for example, \( \text{clipIIIaloe} = \text{calliope} \).

New Problems

13. Find an expression for the perfect shuffle of two languages \( L_1 \) and \( L_2 \) in terms of operations like morphism, inverse morphism, and intersection. Hint: one approach is to modify the expression we found for ordinary shuffle. Conclude that if \( L_1, L_2 \) are regular then so is the perfect shuffle of \( L_1 \) and \( L_2 \).

14. Define

\[
\text{min}(L) = \{ x \in L : \text{no proper prefix of } x \text{ is in } L \}.
\]

Find an expression for \( \text{min}(L) \) in terms of operations like quotient, complement, etc. Conclude that if \( L \) is regular, so is \( \text{min}(L) \).

15. Define

\[
\text{max}(L) = \{ x \in L : x \text{ is not a proper prefix of any } y \in L \}.
\]
Find an expression for \(\max(L)\) in terms of operations like quotient, complement, etc. Conclude that if \(L\) is regular, so is \(\max(L)\).

16. Is it possible to avoid the pattern \(xx'\), where \(x'\) is a conjugate of \(x\), over a 3-letter alphabet? (Remember that “avoid the pattern” means “does there exist an infinite word containing no occurrences of the pattern as a subword?”)

17. Is the Thue-Morse word \(t\) recurrent? That is, if \(x\) is a subword of \(t\), must \(x\) occur at infinitely many different positions of \(t\)?

18. Is the following language regular? \(\{xwx^R : x, w \in \{0, 1\}^+\}\).

19. Is the following language regular? \(\{xx^Rw : x, w \in \{0, 1\}^+\}\).

20. Show that two words \(x, y\) are conjugates iff there exists a word \(t\) such that \(xt = ty\). Furthermore, if \(t\) exists then we can always find one with \(|t| \leq |x|\).

21. A follow-up to #20: show that two words \(x, y\) are conjugates iff there exists a word \(t\) such that \(xt = ty\) or a word \(s\) such that \(sx = ys\). Furthermore argue that there exists such an \(s\) or a \(t\) of length \(\leq |x|/2\).

Older Problems Unsolved from Previous Sessions

2. Show that the number of bordered words of length \(n\) is the same as the number of words of length \(n\) that begin with a nonempty even-length palindrome. Hint: find a bijection. This one is hard!

3. Is the number of bordered words of length \(n\) the same as the number of words of length \(n\) that begin with a nonempty square? Prove or disprove. Hint: do some computation.

4. Show that \(x\) is an even-length palindrome if and only if there exists a string \(y\) such that \(x = y \overline{III} y^R\).

5. Call a language \(L\) commutative if for all \(x, y \in L\) we have \(xy = yx\). Show that \(L\) is commutative if and only if there exists a word \(w\) such that \(L \subseteq w^*\).

9. Prove or disprove: for all integers \(m, n \geq 1\), \(x^n\) is an antipalindrome if and only if \(x^n\) is an antipalindrome.

10. Can you construct an aperiodic infinite word in which there are powers of arbitrarily large exponent beginning at every position? Hint: construct it iteratively.

11. When is the concatenation of two antipalindromes an antipalindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.

12. When is the concatenation of two antipalindromes a palindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.