Recall that a word $x$ is a palindrome if $x = x^R$, where $x^R$ denotes the reversal of $x$.

For a word $x \in \{0,1\}^*$, let $\overline{x}$ denote the word obtained by changing each 0 to 1 and vice versa. Call a word $x$ an antipalindrome if $\overline{x} = x^R$. Thus, for example, 001011 is an antipalindrome.

Recall that two words $x, y$ are conjugates if there are words $u, v$ such that $x = uv$ and $y = vu$.

1. Show that two words $x, y$ are conjugates iff there exists a word $t$ such that $xt = ty$. Furthermore, if $t$ exists then we can always find one with $|t| \leq |x|$.

2. A follow-up: show that two words $x, y$ are conjugates iff there exists a word $t$ such that $xt = ty$ or a word $s$ such that $sx = ys$. Furthermore argue that there exists such an $s$ or a $t$ of length $\leq |x|/2$.

3. Is it possible to avoid the pattern $xx'$, where $x'$ is a conjugate of $x$, over a 3-letter alphabet? (Remember that “avoid the pattern” means “does there exist an infinite word containing no occurrences of the pattern as a subword?”)

4. Can you construct an aperiodic infinite binary word in which there is a square beginning at every position? Here “aperiodic” means “not ultimately periodic”.

5. Prove that a string $x$ is an antipalindrome if and only if there exists $y$ such that $x = y\overline{y}^R$.

6. Is the Thue-Morse word $t$ recurrent? That is, if $x$ is a subword of $t$, must $x$ occur at infinitely many different positions of $t$?

7. When is the concatenation of two antipalindromes an antipalindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.

8. When is the concatenation of two antipalindromes a palindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.