Here’s how these problem-solving sessions work.

Start by working on the problem indicated by the number on the front of this sheet. If you find a solution, let me know and sketch it for me. If it’s correct, you then present it on the board to the class. Next, you write up your solution within one week, in a format I can post for the rest of the class, and send it to me. If you do all that, you fulfill your 10% course mark allocated to these sessions.

If you quickly succeed in solving a problem (some of them are very easy!), try one of the unsolved problems from previous sessions, which are given at the end.

Recall that a word \( x \) is a palindrome if \( x = x^R \), where \( x^R \) denotes the reversal of \( x \).

For a word \( x \in \{0, 1\}^* \), let \( \overline{x} \) denote the word obtained by changing each 0 to 1 and vice versa. Call a binary word \( x \) an antipalindrome if \( \overline{x} = x^R \). Thus, for example, \( 001011 \) is an antipalindrome.

Recall that \( III \) denotes the “perfect shuffle”, so that, for example, \( \text{clip} III \text{aloe} = \text{calliope} \).

**New Problems**

22. Show that if \( M \) is an \( n \)-state DFA, and accepts at least one string that is a palindrome, then it accepts a palindrome of length \( < 2n^2 \). Hint: using \( M \) design an NFA-\( \epsilon \) to accept the first halves of the palindromes in \( L(M) \).

23. Let \( L_1, L_2 \) be regular languages with \( L_2 - L_1 \) infinite. Show there exists a regular language \( L \) with \( L_1 \subseteq L \subseteq L_2 \) with both \( L_2 - L \) and \( L - L_1 \) infinite. (Here \( A - B \) is set difference, defined to be those strings in \( A \) but not in \( B \).)

24. Let \( L \) be a regular language, and let \( s \) be a substitution by regular languages. Must \( s^{-1}(L) = \{ x : s(x) \subseteq L \} \) be regular?
25. Give an example of a unary nonregular language $L$, not containing $\epsilon$, for which $L^2$ is regular.

26. Call a language $L$ bounded if there exist a finite number of words $w_1, w_2, \ldots, w_n$ such that $L \subseteq w_1^* \cdots w_n^*$. Give an example of a regular language that is not bounded.

**Older Problems Unsolved from Previous Sessions**

2. Show that the number of bordered words of length $n$ is the same as the number of words of length $n$ that begin with a nonempty even-length palindrome. Hint: find a bijection. This one is hard!

3. Is the number of bordered words of length $n$ the same as the number of words of length $n$ that begin with a nonempty square? Prove or disprove. Hint: do some computation.

4. Show that $x$ is an even-length palindrome if and only if there exists a string $y$ such that $x = y \mathbb{I} y^R$.

5. Call a language $L$ commutative if for all $x, y \in L$ we have $xy = yx$. Show that $L$ is commutative if and only if there exists a word $w$ such that $L \subseteq w^*$.

9. Prove or disprove: for all integers $m, n \geq 1$, $x^m$ is an antipalindrome if and only if $x^n$ is an antipalindrome.

12. When is the concatenation of two antipalindromes a palindrome? Give necessary and sufficient conditions. Possible strategy: do some experiments.

13. Find an expression for the perfect shuffle of two languages $L_1$ and $L_2$ in terms of operations like morphism, inverse morphism, and intersection. Hint: one approach is to modify the expression we found for ordinary shuffle. Conclude that if $L_1, L_2$ are regular then so is the perfect shuffle of $L_1$ and $L_2$.

14. Define

$$\min(L) = \{ x \in L : \text{no proper prefix of } x \text{ is in } L \}.$$ 

Find an expression for $\min(L)$ in terms of operations like quotient, complement, etc. Conclude that if $L$ is regular, so is $\min(L)$.

16. Is it possible to avoid the pattern $xx'$, where $x'$ is a conjugate of $x$, over a 3-letter alphabet? (Remember that “avoid the pattern” means “does there exist an infinite word containing no occurrences of the pattern as a subword?”)

19. Is the following language regular? $\{xx^Rw : x, w \in \{0, 1\}^+\}$.

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